LCA algorithm

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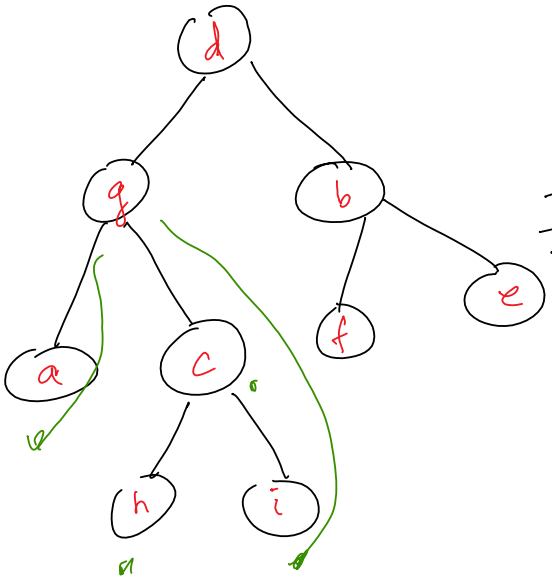
The Lowest Common Ancestor problem



Problem: Finding the **lowest common ancestor** of any two given nodes in the tree T.

Input: Tree T with n nodes
Nodes i & j

Output: lca(i, j)

Goal: Answer lca queries of the kind lca(i, j) in O(1) time

Examples:

lca(a, b) = d
lca(i, a) = g
lca(h, c) = c
lca(b, e) = b

For solving the LCA problem, we will visit Bender-Farach's algorithm (from the late '90's).

Bender, Michael A., and Martin Farach-Colton. "The LCA problem revisited." In *Latin American Symposium on Theoretical Informatics*, pp. 88-94. Springer, Berlin, Heidelberg, 2000.

http://spin2013.cs.sunysb.edu/~bender/newpub/BenderFa00-lca.pdf

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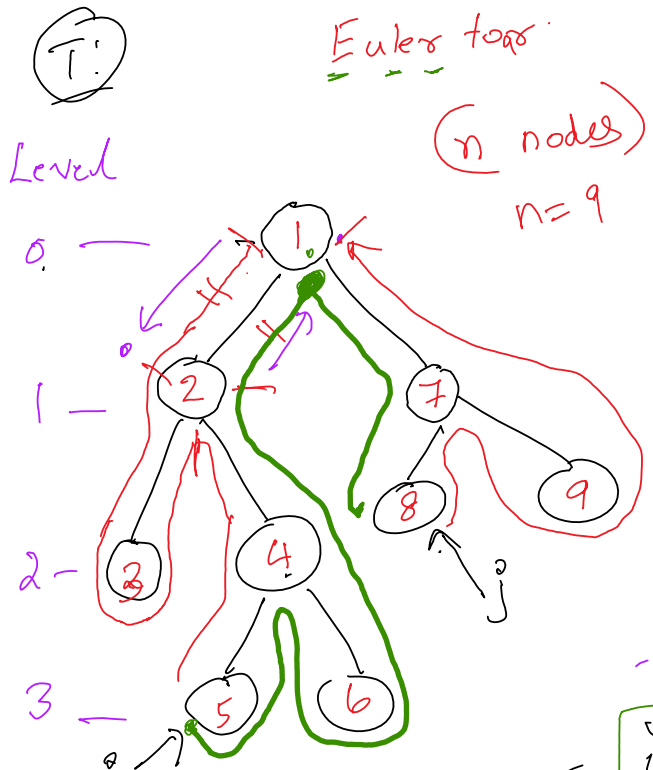
Preprocessing $T \xrightarrow{O(n)} \{E, L, R\} \Rightarrow LCA(i,j)$ $O(1)$ time

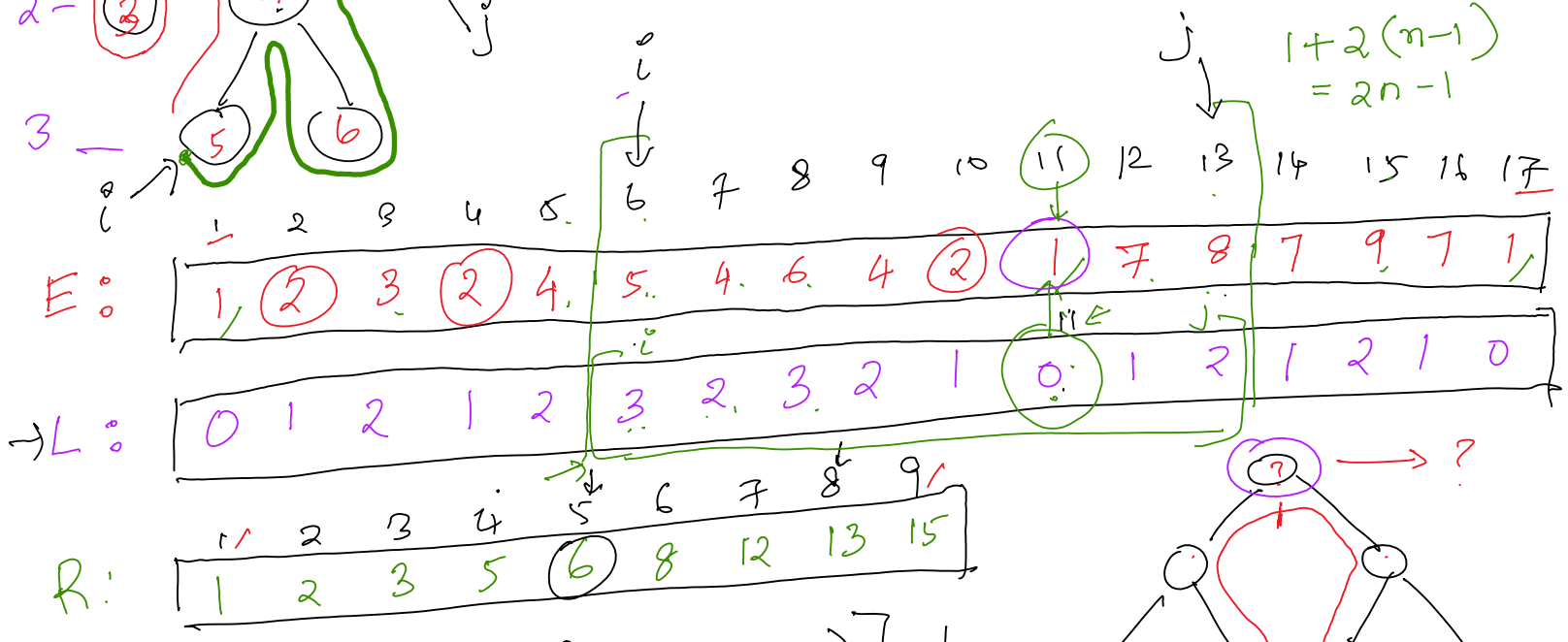
Bender - Farach (2000)

$O(n)$.

1) Euler tour of T
   - → E : Euler tour ← $2n-1$
2) → L : Level array ← $2n-1$
3) → R : Reverse array ← n
   - R[i] ← 1st occurrence of node i in E.



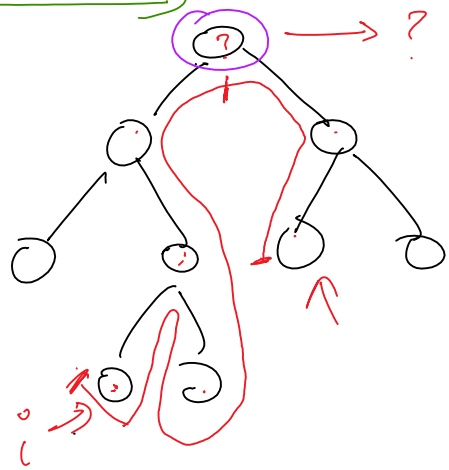
E.g: lca (4, 8)



$lca(i,j) = E[RMQ_L(R[i], R[j])] = 1$

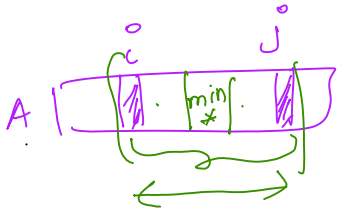
$O(1)$ if $RMQ_L(R[i], R[j])$ is $O(1)$ time

6 13 11



RMQ: Range Minimum Query

$RMQ_A(i,j)$ ← The index of the minimum $A[i \cdots j]$



RMQ function

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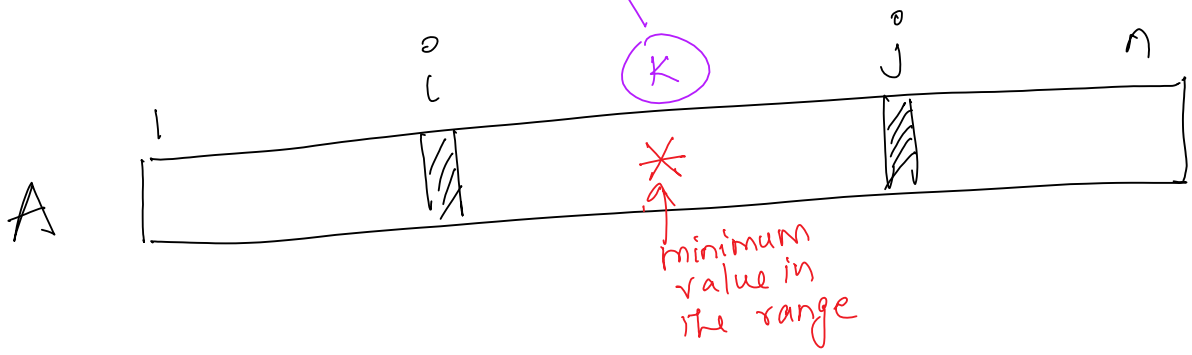
$RMQ_A(i,j)$:

Given as **input**, an array $A[1..n]$ (of ints) and two indices i & j

Return as **output**, the location of the minimum in the array interval $A[i..j]$



Note: It is trivial to be able to answer $RMQ_A(i,j)$ in $O(n)$ time.

Q) Is it possible to answer any RMQ query in $O(1)$ time?

B-prep(A[1..n]): From array A[1..n], generate a set of O(lgn) "B-arrays"

LCA algorithm B-prep(A) ⇒ $\{B_0, B_1, B_2, \cdots B_{\lg n}\}$

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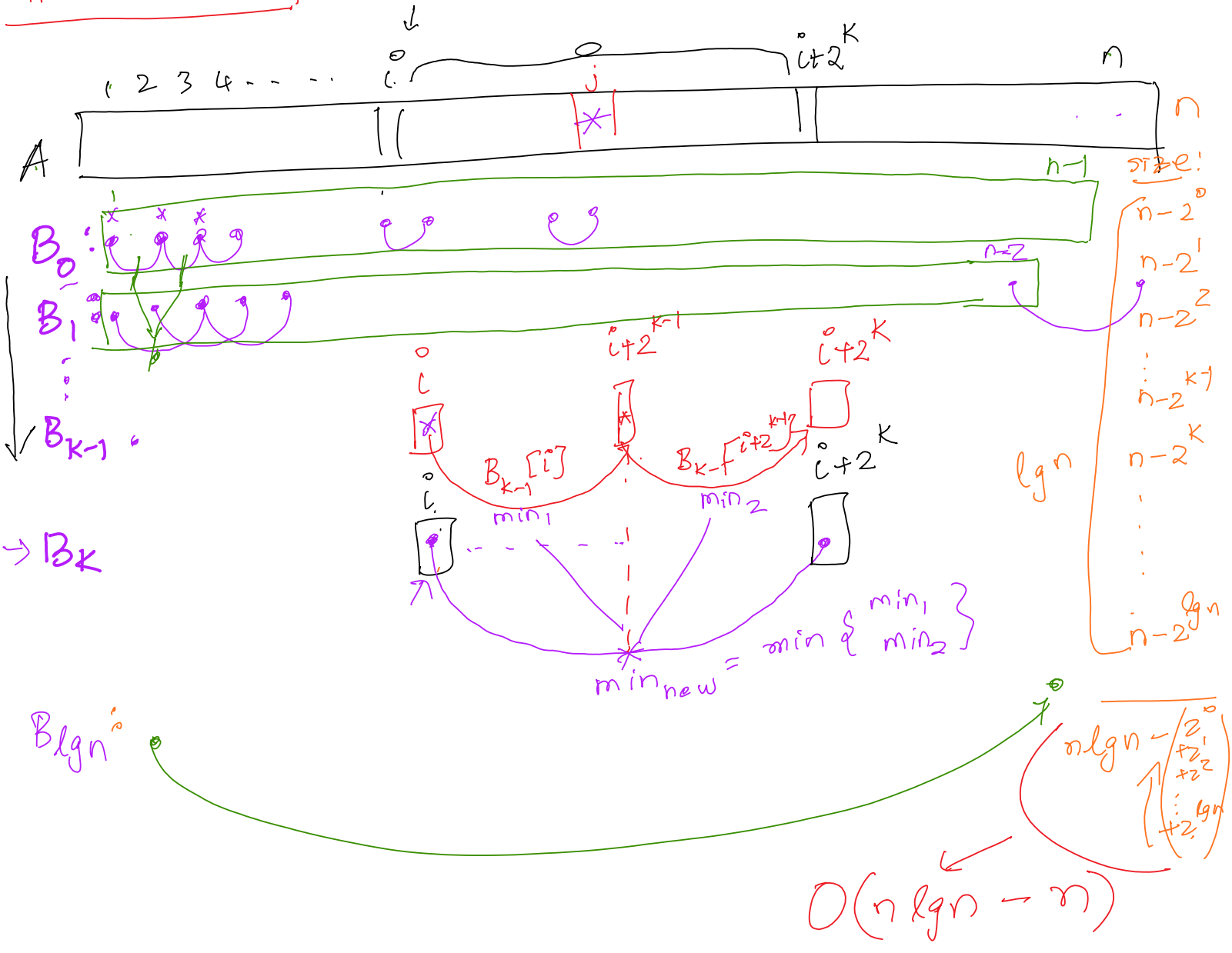
s.t.

$B_K[i] \leftarrow$ minloc(A[i .. $i+2^K$])

$= j$

Let **minloc()**:
→ the location/index of the minimum



B-prep()

Sum of all the sizes of all B-arrays = $O(n \lg n)$

A[1..n] ⇒ $\{B_0, B_1, \cdots B_{\lg n}\}$
$O(n \lg n)$

⇒ Time to construct all B-array = $O(n \lg n)$

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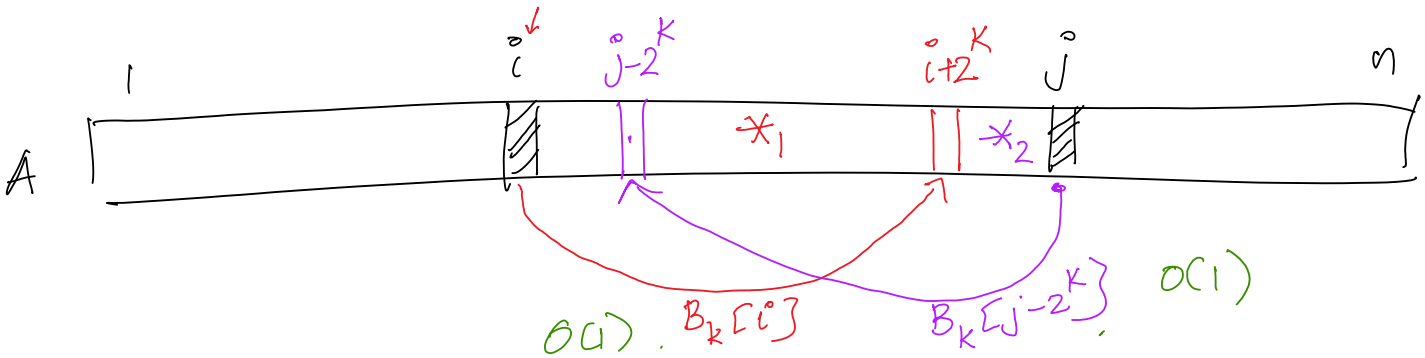
Q) How to answer $RMQ_A(i,j)$ using the B-arrays?

A) Step 1) Preprocessing $A[1..n] \xrightarrow[O(n \lg n)]{}$ B-arrays $\{B_0, B_1, \dots B_{\lg n}\}$

Step 2) Querying $RMQ_A(i,j)$:



$O(1)$ $B_k[i]$ $B_k[j-2^k]$ $O(1)$

$*_{(i,j)} \leftarrow \text{minloc}\{*_1, *_2\}$ $O(1)$

$K = ?$ $\quad 2^K \le j - i$

$K \le \lg(j-i)$

$O(1) \rightarrow K = \lfloor \lg(j-i) \rfloor$

$A[1..n] \Rightarrow$ B-prep(A) $\Rightarrow \{B_0, B_1, \dots, B_{\lg n}\} \Rightarrow$ Query time

$RMQ_A(i,j)$ ⇓ lookup B_K values & compare

$O(n \lg n)$ time ↗ ? ↘ $O(n)$ time

$O(1)$ time

$O(1)$ time

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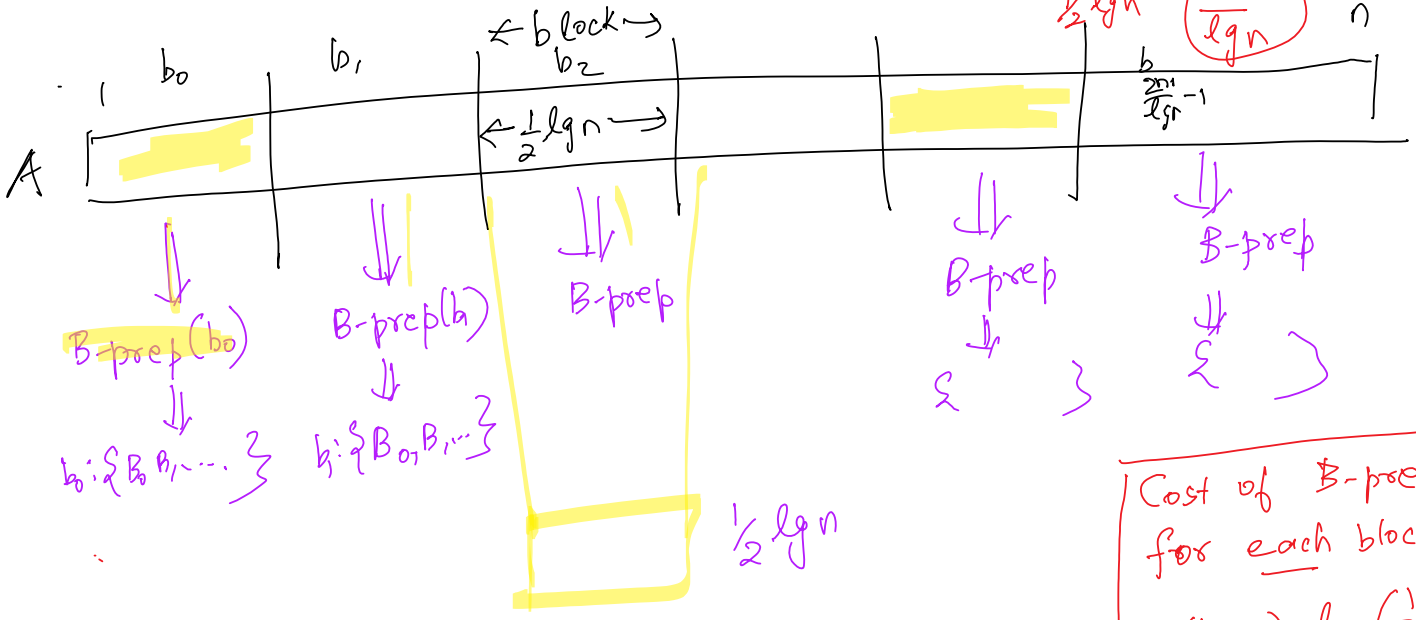
Q) Can we bring down the cost of preprocessing from $O(n \lg n) \Longrightarrow O(n)$ possibly?

A) Let us divide $A[1..n]$ into "blocks" of size $\frac{1}{2}\lg n$ each.

$\Rightarrow$ #blocks $= \frac{n}{\frac{1}{2}\lg n} = \frac{2n}{\lg n}$



Cost of B-prep() for each block

$= \left(\frac{1}{2}\lg n\right) \lg\left(\frac{1}{2}\lg n\right)$

$= O\left(\frac{1}{2}\lg n \lg(\lg \sqrt{n})\right)$

#blocks $= \frac{2n}{\lg n}$

Total cost

$= \frac{2n}{\lg n} \times \frac{1}{2}\lg n \lg \lg \sqrt{n}$

$= O(n \lg \lg \sqrt{n})$

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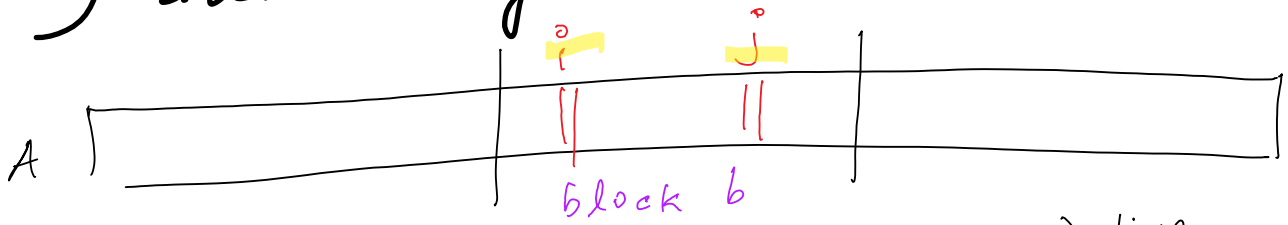
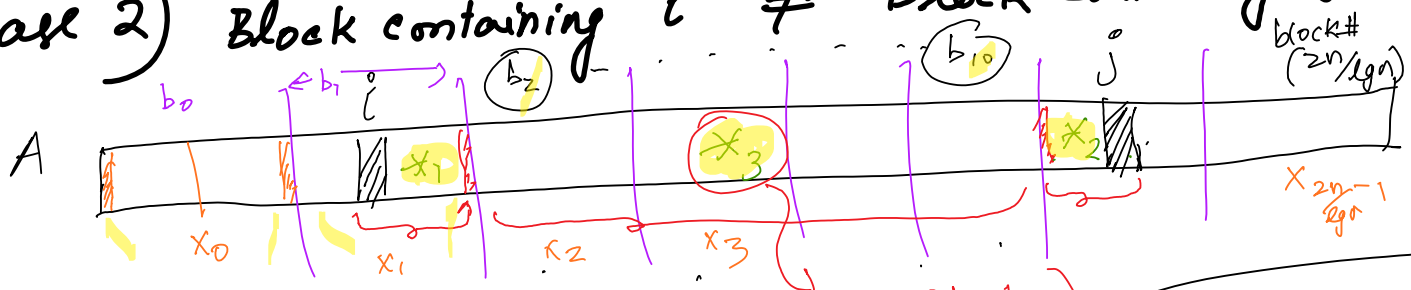
Two cases for RMQ(i,j) queries:

Case 1) Block containing i° = Block containing j°



$\Rightarrow$ RMQ(i°,j°) can be answered in O(1) time using B-arrays for block b

Case 2) Block containing i° ≠ Block containing j°

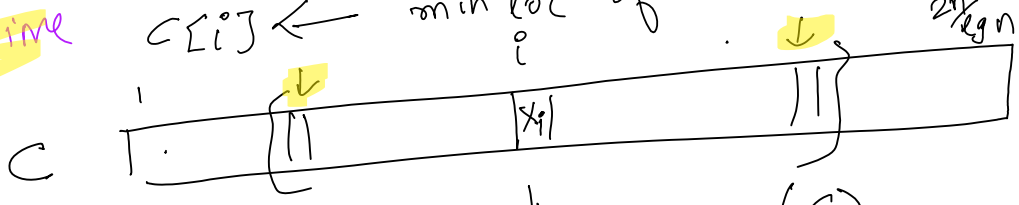


$\Rightarrow$ To compute $*_3$:

1) Build an array $C[1 \ldots \frac{2n}{\lg n}]$ s.t

$C[i] \leftarrow$ min loc of block b_i

$O(\frac{2n}{\lg n})$ time



2) B-prep(C):

$\Downarrow$ B-prep(C)

generate all B-arrays for C in O(1) time.

$\Rightarrow$ RMQ queries on C array

cost of B-prep(C) = $\left(\frac{2n}{\lg n}\right) \lg\left(\frac{2n}{\lg n}\right) = O\left(\frac{2n}{\lg n} \lg n\right)$

$= O(n)$ time

$RMQ_A(i,j) = \min_{\text{loc}} \{*_1, *_2, *_3\}$

$\Rightarrow O(1)$ time

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Overall Preprocessing Algorithm so far:

Input: $A[1..n]$

Time Complex:

PREPROCESSING

1) Partition into $\frac{2n}{\lg n}$ blocks

1a) B-prep (each block separately) } $O(n \lg \lg n)$

2) Generate C-array

2b) B-prep(C) } $O(n)$

3) Answer $RMQ_A(i,j)$ queries } $O(1)$ time

$\Rightarrow$ $A[1..n]$ (input) $\Rightarrow$ [Preprocessing] $\Rightarrow$ [RMQ queries] $\circlearrowright (i,j)$

Preprocessing: $O(n \lg \lg n)$

RMQ queries: $O(1)$ time per query

Q) Can we make preprocessing time $O(n)$?

Idea

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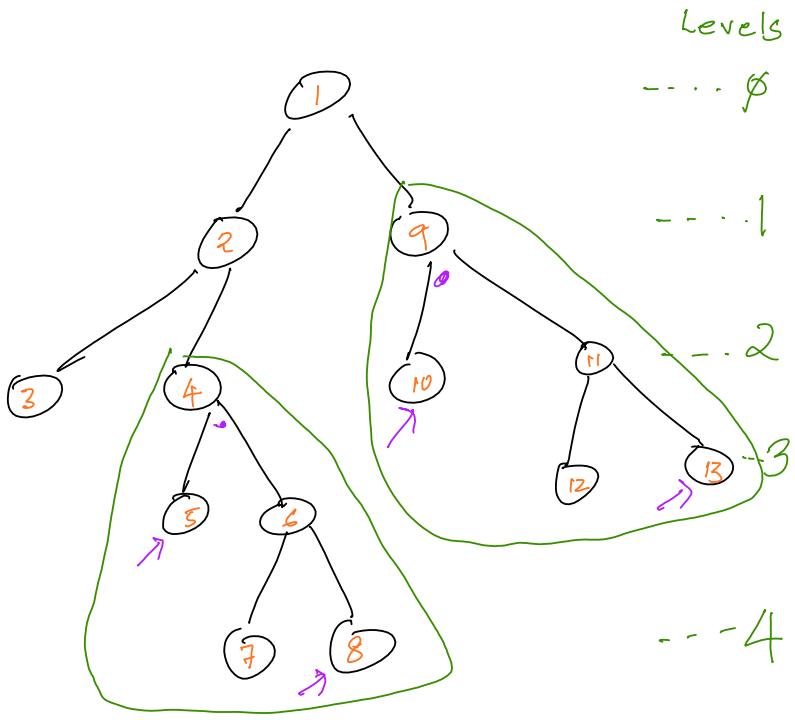

E array:

1 2 3 2 4 5 4 6 7 6 8 6 4 2 1 9 10 9 11 12 11 13 11 9 1

L array:

0 1 2 1 2 3 2 3 4 3 4 3 2 1 0 1 2 1 2 3 2 3 2 1 0

B-prep

0 +1 -1 +1 +1 -1 +1 -1 -1

B-prep X

0 +1 -1 +1 +1 -1 +1 -1 -1

$(\frac{1}{2}\lg n - 1)$ bits

block

$\frac{1}{2}\lg n - 1$

0 | ±1 | ±1 | ±1 | . . . . | ±1

$\frac{2n}{\lg n}$ blocks $\geq \sqrt{n}$

#total distinct fingerprints $= 2^{(\frac{1}{2}\lg n - 1)} = 2^{\lg_2 \sqrt{n} - 1}$

$\leq \sqrt{n} \times (\frac{1}{2}\lg n)$ lglg$\sqrt{n}$ $= \frac{2^{\lg \sqrt{n}}}{2} = \frac{\sqrt{n}}{2}$

$= O(\sqrt{n})$

Cost of B-prep all distinct fingerprints $= O(n)$ time $= O(\sqrt{n} \times \sqrt{n})$

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Key Observation: Since the target array for preprocessing is the **level (L)** array, we can take advantage of the following property of L:

⇒ Every pair of consecutive elements in L differs by 1. // because of the Euler tour property

⇒ Each block generated from the L-array can be represented as a +1/-1 array (offset by the first element)

e.g., block X:

4	5	6	5	4	3	4	5	4	5

⇒

0	+1	+1	-1	-1	-1	+1	+1	-1	+1

⇒ let us call this pattern the block's "fingerprint"

Now, two blocks may look different but may have the same fingerprint.

e.g., block Y:

8	9	10	9	8	7	8	9	8	9

⇒

0	+1	+1	-1	-1	-1	+1	+1	-1	+1

⇒ same as block X's fingerprint

Two blocks with the same fingerprint will have their minimums located at the same index. (as shown highlighted above).

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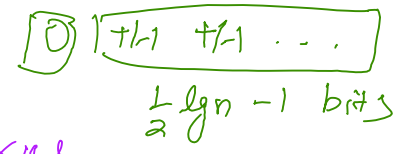
Idea: Instead of B-prep() each block of L separately, (a) Identify and bin blocks by their fingerprints

(b) B-prep() all the fingerprints.

Q) What is the maximum number of distinct fingerprints possible for the blocks derived from an array of size n?

Each block is of size $\left(\frac{1}{2}\lg n\right)$

$\Rightarrow$ Each block represents a binary pattern $(+/-1)$ of length $\left(\left(\frac{1}{2}\lg n\right)-1\right)$



$\Rightarrow$ Max # distinct binary patterns

$$= 2^{\frac{1}{2}\lg n - 1} = \frac{2^{\lg \sqrt{n}}}{2} = \frac{\sqrt{n}}{2}$$

$\therefore$ Cost of B-prep all distinct block fingerprints

$$= O\left(\frac{\sqrt{n}}{2} \times \left(\frac{1}{2}\lg n\right) \log\left(\frac{1}{2}\lg n\right)\right)$$

$$= O\left(\sqrt{n} \lg n \lg(\lg \sqrt{n})\right)$$

$$= O(n)$$

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Overall Algorithm :

Input: $A[1..n]$

Time Complex:

PREPROCESSING

1) Partition into $\frac{2n}{\lg n}$ blocks

1a) B-prep (each block fingerprint separately) } $O(n)$

2) Generate C-array

2b) B-prep(C) } $O(n)$

3) Answer $RMQ_A(i,j)$ queries } $O(1)$ time

$\Rightarrow$ $A[1..n]$ (input) $\Rightarrow$ [Preprocessing] $\Rightarrow$ [RMQ queries] $y(i,j)$

$\Theta(n)$

$O(1)$ time per query

LCA algorithm

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Summary of the main algorithmic results:

	Time complexity: Preprocessing	Querying
$RMQ_A(i,j)$ for an arbitrary array of size n	$O(n \lg \lg \sqrt{n})$	$O(1)$
$RMQ_A(i,j)$ for a binary array of size n	$O(n)$	$O(1)$
$lca(i,j)$ for a tree with n nodes	$O(n)$	$O(1)$
$lcp(suff_i, suff_j)$ for a string of size n	$O(n)$	$O(1)$