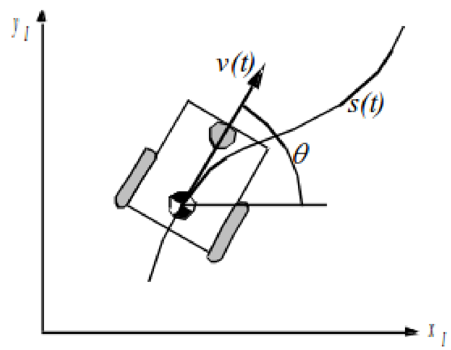
Kinematics

- So far we have looked at different kinds of motion in a qualitative way.
- One way to program robots to move is trial and error.
- A somewhat better way is to establish mathematcially how the robot *should* move, this is *kinematics*.
- Rather kinematics is the business of figuring how a robot will move if its motors work in a given way.
- Inverse-kinematics then tells us how to move the motors to get the robot to do what we want.
- We'll look at two tiny bits of the kinematics world.

A formal model

- We will assume, as people usualy do, that the robot's location is fixed in terms of three coordinates:

$$(x_I, y_I, \theta)$$



- Given that the robot needs to navigate to a location (x_G, y_G, θ_G), it can determine how x, y and θ need to change.
  - BUT it can't control these directly.

- All it has access to are the speeds of its wheels:

$$\dot{\varphi}_1, \ldots \dot{\varphi}_n$$

The steering angle of the steerable wheels:

$$\beta_1, \ldots \beta_m$$

and the speed with which those steering angles are changing.

$$\dot{\beta}_1, \ldots \dot{\beta}_m$$

- Together these determine the motion of the robot:

$$f(\dot{\varphi}_1, \ldots \dot{\varphi}_n, \beta_1, \ldots \beta_m, \dot{\beta}_1, \ldots \dot{\beta}_m) = \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta} \end{bmatrix}$$

- *Forward kinematics*

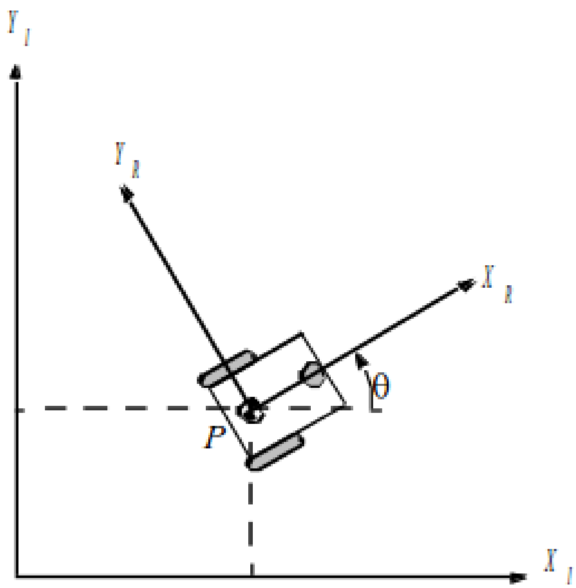
- This is not what we want.
- But we can get what we want from it:

$$\begin{bmatrix} \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_n \\ \beta_1 \\ \vdots \\ \beta_m \\ \dot{\beta}_1 \\ \vdots \\ \dot{\beta}_m \end{bmatrix} = f(\dot{x}_I, \dot{y}_I, \dot{\theta})$$

- *Reverse kinematics*

Representing robot position

- The robot knows how it moves relative to its center of rotation.
- This is not the same as knowing how it moves relative to the world.



Two systems of coordinates:

- Initial Frame: $\{X_I, Y_I\}$
- Robot Frame: $\{X_R, Y_R\}$

- Robot position:

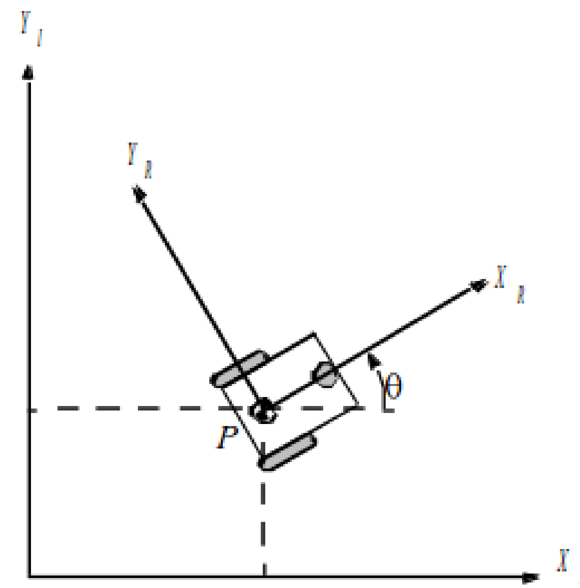
$$\xi_I = [x_I, y_I, \theta_I]^T$$

- Mapping between frames:

$$\begin{aligned} \dot{\xi}_R &= R(\theta)\dot{\xi}_I \\ &= R(\theta)\left[\dot{x}_I, \dot{y}_I, \dot{\theta}_I\right]^T \end{aligned}$$

where

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- In other words

$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix}$$

meaning that:

$$\begin{aligned} \dot{x}_R &= \dot{x}_I \cos(\theta) + \dot{y}_I \sin(\theta) + \dot{\theta}_I.0 \\ \dot{y}_R &= -\dot{x}_I \sin(\theta) + \dot{y}_I \cos(\theta) + \dot{\theta}_I.0 \\ \dot{\theta}_R &= \dot{x}_I.0 + \dot{y}_I.0 + \dot{\theta}_I.1 \end{aligned}$$

- This is the forward kinmatic model, but this still isn't what we want — we want the reverse kinematic model:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

  where $R(\theta)^{-1}$ is the inverse of $R(\theta)$.

- Often $R(\theta)^{-1}$ is hard to compute, but luckily for us in this case it isn't.

- We have:

$$R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  which we can use to establish $\dot{x}_I$, $\dot{y}_I$, $\dot{\theta}_I$

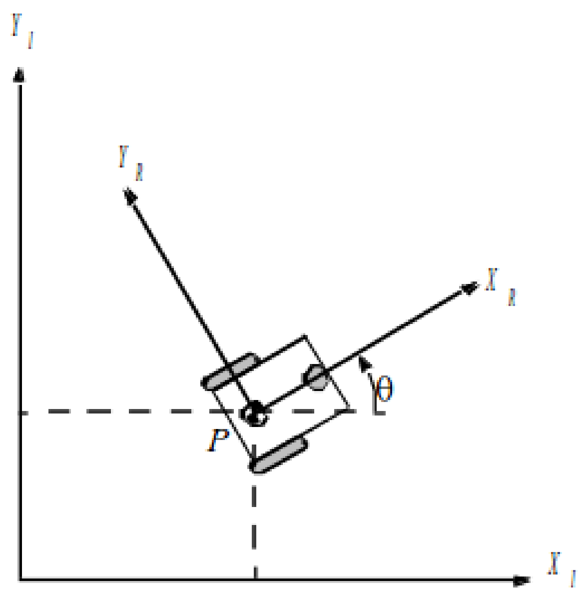
- To do this we compute:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = \boldsymbol{R}(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

meaning that:

$$\begin{aligned} \dot{x}_I &= \dot{x}_R \cos(\theta) - \dot{y}_R \sin(\theta) + \dot{\theta}_R.0 \\ \dot{y}_I &= \dot{x}_R \sin(\theta) + \dot{y}_R \cos(\theta) + \dot{\theta}_R.0 \\ \dot{\theta}_I &= \dot{x}_R.0 + \dot{y}_R.0 + \dot{\theta}_R.1 \end{aligned}$$

Down to the structure of the robot

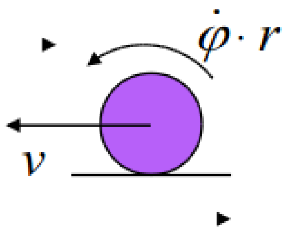


- We can now identify the motion of the robot, in the global frame, if we know:

$$\dot{x}_R, \dot{y}_R, \dot{\theta}$$

but how do we tell what these are?

- We compute it from what we can measure like the speed of the wheels:



- Some assumptions — *constraints* on the motion of the robot:
  - Movement on a horizontal plane
  - Point contact of the wheels; wheels not deformable
  - Pure rolling, so $v = 0$ at contact point; no slipping, skidding or sliding
  - No friction for rotation around contact point
  - Steering axes orthogonal to the surface
  - Wheels connected by rigid frame (chassis)

- Consider differential drive.
- Wheels rotate at $\dot{\varphi}$
- Each wheel contributes:

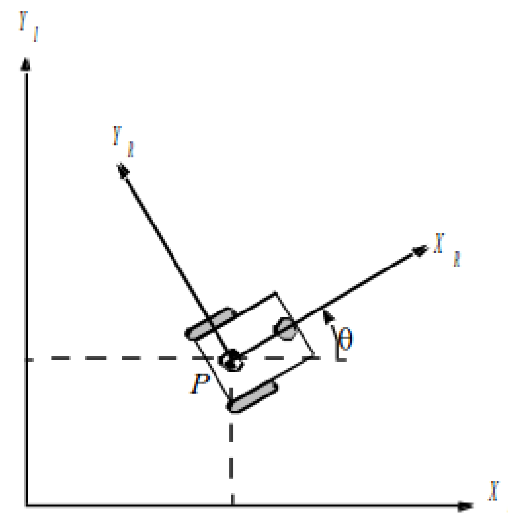
$$\frac{r\dot{\varphi}}{2}$$

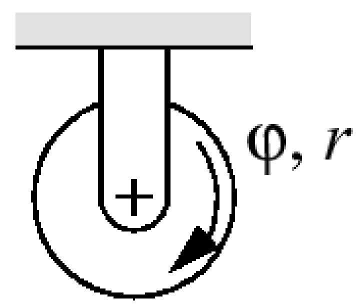
  to motion of center of rotation.
- Total speed is the sum of two contributions.
- Rotation due to right wheel is

$$\omega_r = \frac{r\dot{\varphi}}{2l}$$

  counterclockwise about the left wheel.

  l is the distance between wheels.





- Rotation due to the left wheel is:

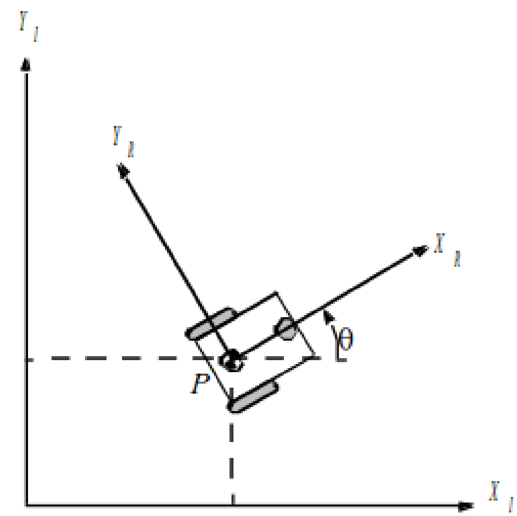
$$\omega_l = \frac{-r\dot{\varphi}}{2l}$$

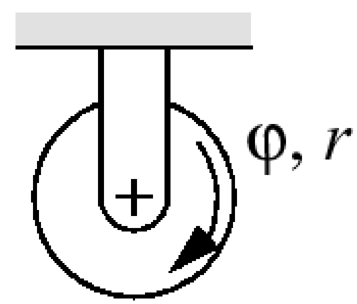
counterclockwise about the right wheel.

- Combining these components we have:

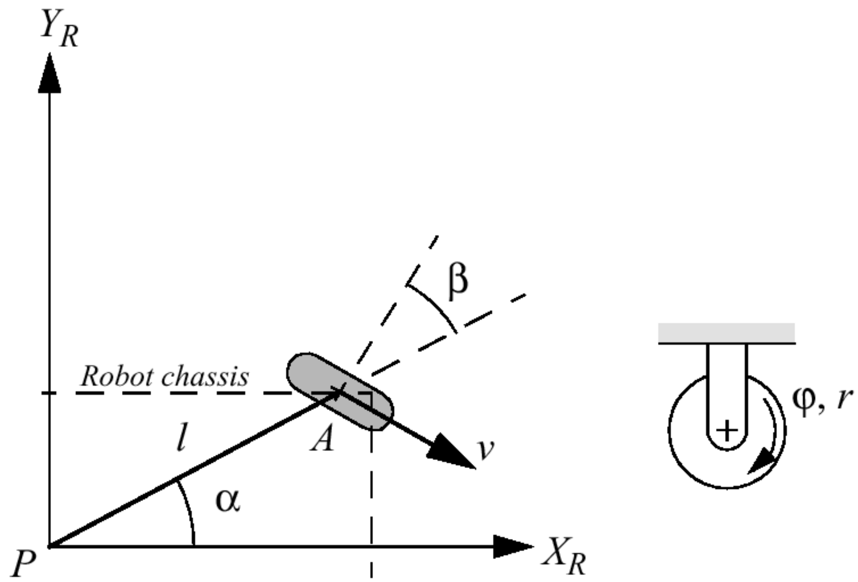
$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = \begin{bmatrix} \frac{r\dot{\varphi}_r}{2} + \frac{r\dot{\varphi}_l}{2} \\ 0 \\ \frac{r\dot{\varphi}_r}{2l} - \frac{r\dot{\varphi}_l}{2l} \end{bmatrix}$$

- And we can combine these with $R(\theta)^{-1}$ to find motion in the global frame.



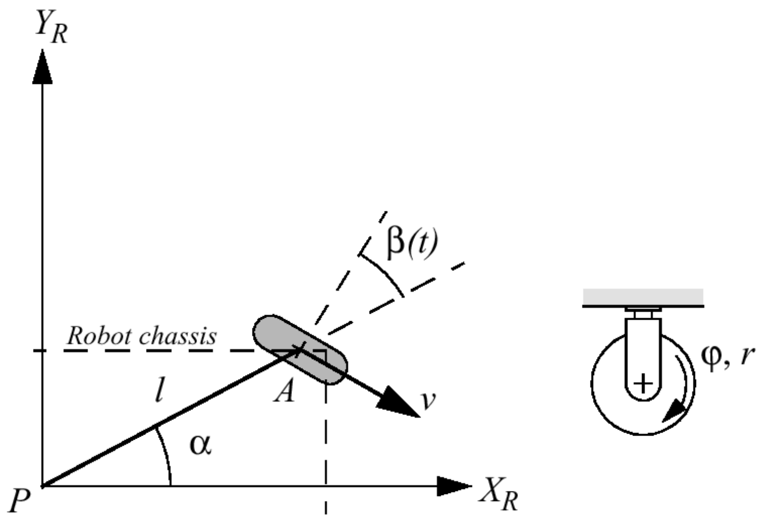


Wheel geometry

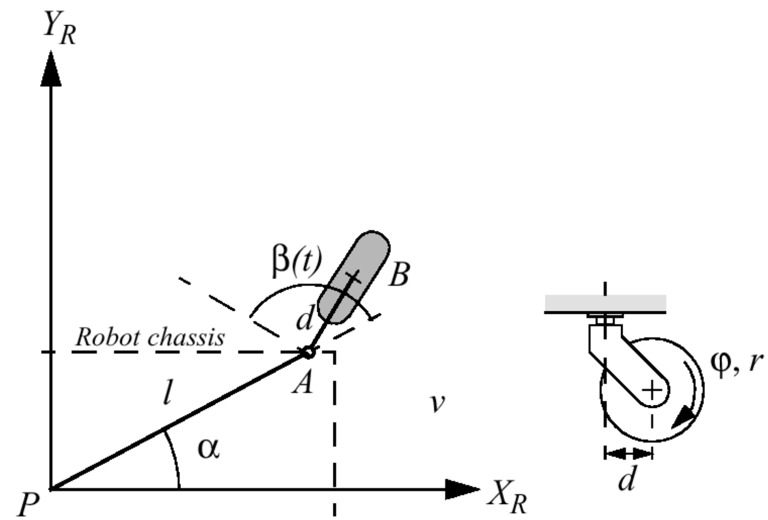


- Making sure the assumptions hold imposes constraints.
- For example, ensuring that the wheel doesn't slip says something about motion in the direction of the wheel.

More complex scenarios



Steered standard wheel

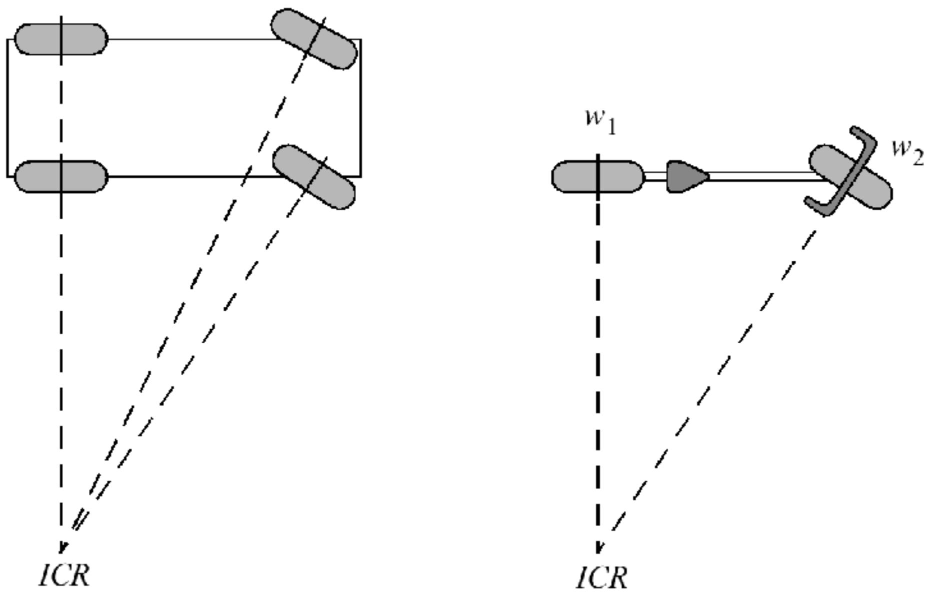


Caster

- More parameters.
- Only fixed and steerable standard wheels impose constraints.

Robot mobility

- The sliding constraint means that a standard wheel has no lateral motion.
  - Zero motion line through the axis.



Instantaneous center of rotation

- Has to move along a circle whose center is on the zero motion line.

- A differential drive robot has just one line of zero motion.

- Thus its rotation is not constrained
  - It can move in any circle it wants.
- Makes it very easy to move around.
- This depends on the number of independent kinematic constraints.

- Formally we have the notion of a *degree of mobility*

$$\delta_m = 3 - \text{number of independent kinematic constraints}$$

- This number is also the number of *independent* fixed or steerable standard wheels.
- The independence is important.
- Differential drive has two standard wheels, but they are on the same axis.
  - So not independent.
- So $\delta_m = 2$ for a differential drive robot
  - Can alter $\dot{x}$ and $\dot{\theta}$ just through wheel velocity.
- A bicycle has two independent wheels, so $\delta_m = 1$
  - Can only alter $\dot{x}$ using wheel velocity.

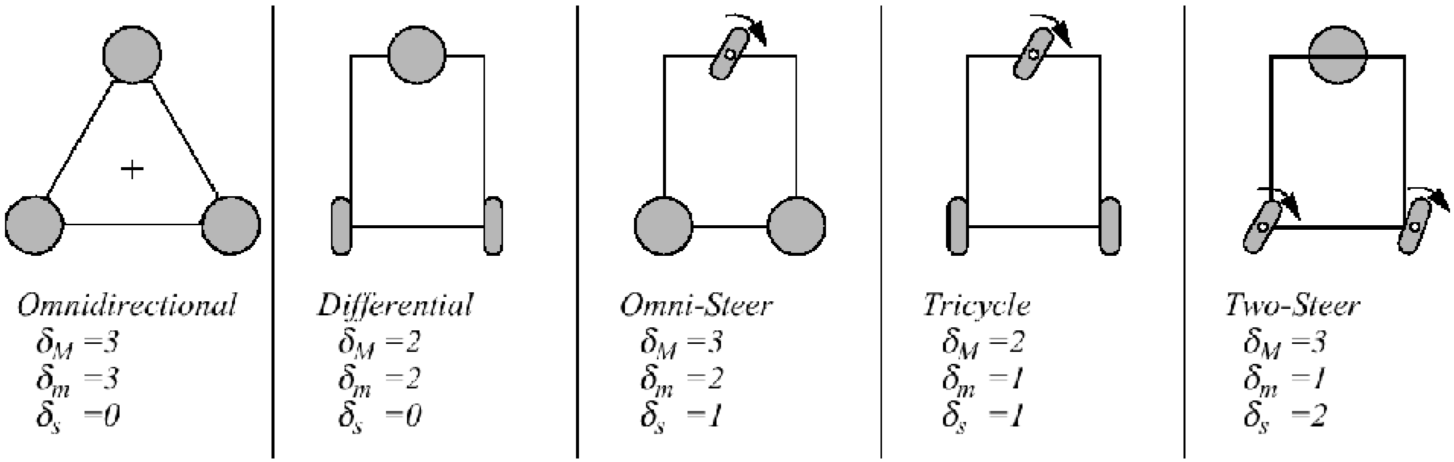
Steerability and Maneuverability

- Steering has an impact on how the robot moves.
- The *degree of steerability* δ_s is then the number of independent steerable wheels.
- Note that a steerable standard wheel will both reduce the degree of mobility and increase the degree of steerability.
- The *degree of maneuverability* is:

$$\delta_M = \delta_m + \delta_s$$

- δ_M tells us how many degrees of freedom a robot can manipulate.
- Two robots with the same δ_M aren't necessarily equivalent (see on).

Common configurations



Holonomy

- Consider a bicycle, it has a $\delta_M = 2$ yet can position itself anywhere in the plane.
  - Workspace is the set of possible configurations.
  - Workspace DOF is 3 in the general case.
- A bicycle only has one DOF that it can control directly.
  - *Differential* DOF
  - DDOF is always equal to δ_m
- A general inequality:

$$DDOF \leq \delta_M \leq DOF$$

- A robot with $DDOF = DOF$ is called *holonomic*

Summary

- This lecture started by looking at locomotion.
- It discussed many of the kinds of motion that robots use, giving examples.
- We then looked a bit at kinematics
  - The business of relating what robots do in the world to what their motors need to be told to do.
- We did a little math, but most of the discussion was qualitative.
- The book goes more into the mathematical detail of establishing kinematic constraints.