Computer Arithmetic Algorithms

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Number Representations

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Information

- Textbook
  - Israel Koren, “Computer Arithmetic Algorithms,” Prentice Hall, 1993.

Outline

- Binary number system
- Radix conversion
- Negative numbers
  - Signed-magnitude
  - One's complement
  - Two's complement
- Addition and subtraction
- Arithmetic shift operations (sign extension)

The Binary Number System

- A binary number of length n

$$(x_{n-1}, x_{n-2}, \dots, x_1, x_0)$$

  - x_i: bit ($\in \{0,1\}$)

- An n-digit binary number

$$X = x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0 = \sum_{i=0}^{n-1} x_i \boxed{2}^i$$

radix

  - $x_i \in \{0,1\}$

- Example
  - $1011_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 11$

The Binary Number System

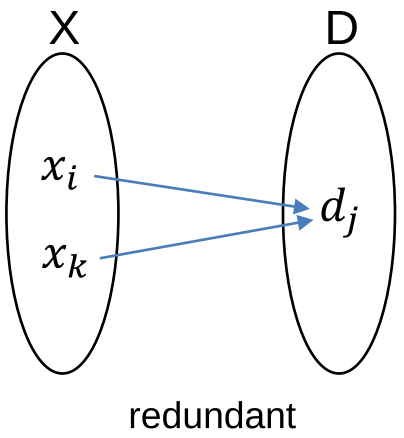
- Radix-10 numbers: decimal numbers
  - $(101)_{10} = 101$
- Radix-2 numbers: binary numbers
  - $(101)_2 = 5$
- Range of representable numbers

$$[X_{min}, X_{max}]$$

  - Example: The range of 4-bit unsigned binary numbers: [0, 15]
- Overflow
  - An arithmetic operation resulting in a number larger than X_{max} or smaller than X_{min}

The Binary Number System

- The conventional number systems
  - ***Nonredundant***
    - Every number has a unique representation.
  - ***Weighted***
    - $X = \sum_{i=0}^{n-1} x_i w_i$
    - w_i: weight
  - ***Positional***
    - w_i depends only on the position of the digit x_i.
    - $w_i = r^i$
    - $0 \leq x_i \leq r - 1$
      - Otherwise (if $x_i \geq r$), the system becomes redundant.



The Binary Number System

- A mixed number (*fixed-point* representation)

$$(\underbrace{x_{k-1}\ x_{k-2}\ \ldots\ x_1\ x_0}_{\text{integral part}} . \underbrace{x_{-1} x_{-2}\ \ldots x_{-m}}_{\text{fractional part}})_r$$

$$= (x_{k-1} r^{k-1} + x_{k-2} r^{k-2} + \cdots + x_1 r^1 + x_0 r^0) + (x_{-1} r^{-1} + \cdots + x_{-m} r^{-m})$$

$$= \sum_{i=-m}^{k-1} x_i r^i$$

- r^{-m}: indicates the position of the radix point.

- *ulp*: unit in the last position
  - $ulp = r^{-m}$

Radix Conversion

- Given
  - $X = X_I + X_F = \sum_{i=0}^{k-1} x_i r^i + \sum_{i=-m}^{-1} x_i r^i$
    - X_I: integral part
    - X_F: fractional part
  - r_D: destination number system

- Integral part

$$X_I = x_0 + r_D(x_1 + r_D(x_2 + \cdots + r_D(x_{k-2} + r_D x_{k-1})))$$
$$0 \leq x_i < r_D$$

  - $x_0 = X_I \, mod \, r_D$
  - $q_0 = x_1 + r_D(x_2 + r_D(x_3 + \cdots))$
  - $x_1 = q_0 \, mod \, r_D$
  - ...

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Radix Conversion

- Fractional part

$$X_F = r_D{}^{-1}(x_{-1} + r_D{}^{-1}(x_{-2} + \cdots + r_D{}^{-1}(x_{-(m-1)} + r_D{}^{-1}x_{-m})))$$
$$0 \le x_i < r_D$$

  - x_{-1}: the integral part of $X_F r_D$
  - $m_{-1} = r_D{}^{-1}(x_{-2} + \cdots + r_D{}^{-1}(x_{-(m-1)} + r_D{}^{-1}x_{-m}))$
  - x_{-2}: the integral part of $m_{-1} r_D$
  - ...

- Example
  - $46.375_{10} = 101110.011_2$

Negative Numbers: Signed-Magnitude

- Representation

$$(x_s\ x_{n-2}x_{n-3}\ \ldots x_1x_0)$$

sign magnitude

- Positive numbers: 0xxxxx
  - $0xxxxx_2$
- Negative numbers: (r-1)xxxxx
  - $1xxxxx_2$

- Range of positive numbers for $(s\ x_{k-2}x_{k-3}\ \ldots x_1x_0 \,.\, x_{-1}x_{-2}\ \ldots x_{-m})$
  - [0, $r^{k-1} - ulp$]
- Range of negative numbers for $(s\ x_{k-2}x_{k-3}\ \ldots x_1x_0 \,.\, x_{-1}x_{-2}\ \ldots x_{-m})$
  - [$-(r^{k-1} - ulp)$, 0]

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Negative Numbers: Signed-Magnitude

- Problems
  - Two zeros
    - 0000...0
    - 1000...0
  - Difficulties in subtraction
    - If X>Y, X-Y is ok.
    - However, Y-X is -(X-Y), so we should compute X-Y, then set the MSB to 1.

Negative Numbers: Complement

- Represent –Y by R – Y.
- Then, –(–Y) = R – (R – Y) = Y.
- If X>Y>0, Y – X = Y + (R – X) = R – (X – Y) = X – Y.

- Two requirements
  - When X>Y>0, X – Y = X + (R – Y) = R + (X – Y) should be X – Y.
  - Computation of R – Y should be simple.

- Notation

$$\overline{x_i} = (r-1) - x_i$$
$$\bar{X} = (\overline{x_{k-1}}, \overline{x_{k-2}}, \dots, \overline{x_0}, \overline{x_{-1}}, \dots, \overline{x_{-m}})$$
$$X + \bar{X} = ((r-1), (r-1), \dots, (r-1))$$
$$X + \bar{X} + ulp = r^k$$

Negative Numbers: Complement

- One's complement (Diminished-radix complement)
  - $R = r^k - ulp$
  - $-X = R - X = \bar{X}$
  - When X>Y>0, $Y - X = Y + (R - X) = R - (X - Y) = -(Y - X)$
  - When X>Y>0, $X - Y = X + (R - Y) = R + (X - Y) = r^k + (X - Y) - ulp = (X - Y) - ulp$
    - This requires a correction step (add *ulp*).
  - When X>0, Y>0, $-X - Y = (R - X) + (R - Y) = R + R - (X + Y) = r^k + R - (X + Y) = R - (X + Y) = -(X + Y)$

- Two's complement (Radix complement)
  - $R = r^k$
  - $-X = R - X = \bar{X} + ulp$
  - When X>Y>0, $Y - X = Y + (R - X) = R - (Y - X) = -(Y - X)$
  - When X>Y>0, $X - Y = X + (R - Y) = R + (X - Y) = r^k + (X - Y) = X - Y$
  - When X>0, Y>0, $-X - Y = (R - X) + (R - Y) = R + R - (X + Y) = r^k + R - (X + Y) = R - (X + Y) = -(X + Y)$

Two's Complement Representation

- One zero
  - 000...0
- A negative number that does not have a positive equivalent
  - 100...0
- Range
  - $[-2^{n-1}, 2^{n-1} - ulp]$

- $X = -x_{n-1} \cdot 2^{n-1} + \sum_{i=0}^{n-2} x_i 2^i$

  Proof) If $x_{n-1} = 0$ (positive), $X = \sum_{i=0}^{n-2} x_i 2^i = -x_{n-1} \cdot 2^{n-1} + \sum_{i=0}^{n-2} x_i 2^i$.

  If $x_{n-1} = 1$ (negative), $X = -[\bar{X} + ulp] = -[\sum_{i=0}^{n-1} \bar{x_i} 2^i + 1]$

  $= -\left[\sum_{i=0}^{n-1} (1 - x_i) 2^i + 1\right] = -\left[2^n - 1 - \sum_{i=0}^{n-1} x_i 2^i + 1\right]$

  $= -2^n + \sum_{i=0}^{n-1} x_i 2^i = -2^{n-1} + \sum_{i=0}^{n-2} x_i 2^i$

One's Complement Representation

- Two zeros
  - 000...0
  - 111...1

- Range
  - $[-(2^{n-1} - ulp), (2^{n-1} - ulp)]$

- $X = -x_{n-1}(2^{n-1} - ulp) + \sum_{i=0}^{n-2} x_i 2^i$

  Proof) If $x_{n-1} = 0$ (positive), $X = \sum_{i=0}^{n-2} x_i 2^i = -x_{n-1} \cdot (2^{n-1} - ulp) + \sum_{i=0}^{n-2} x_i 2^i$.

  If $x_{n-1} = 1$ (negative), $X = -\bar{X} = -\sum_{i=0}^{n-1} \overline{x_i} 2^i = -\sum_{i=0}^{n-1} (1 - x_i) 2^i$

  $= -\left[2^n - 1 - \sum_{i=0}^{n-1} x_i 2^i\right] = -2^n + 1 + \sum_{i=0}^{n-1} x_i 2^i$

  $= -2^{n-1} + 1 + \sum_{i=0}^{n-2} x_i 2^i = -x_{n-1} \cdot (2^{n-1} - ulp) + \sum_{i=0}^{n-2} x_i 2^i$

Addition and Subtraction

- Signed-magnitude

```
  0 1 0 1 1 (11)        0 1 0 1 1 (11)       1 0 0 1 1 (-3)       1 1 0 0 0 (-8)
+ 0 0 0 1 0 (2)       + 0 0 1 1 0 (6)      + 1 0 1 0 1 (-5)     + 1 1 0 0 1 (-9)
= 0 1 1 0 1 (13)      = 1 0 0 0 1 (-1)     =1 0 1 0 0 0 (+8)    =1 1 0 0 0 1 (-1)
                        overflow             overflow             overflow
```

```
  0 1 1 0 1 (13)          1 1 1 0 1 (-13)
+ 1 1 0 0 0 (-8)        + 0 1 0 0 0 (8)
= 1 0 0 1 0 1 (5)       = 1 0 0 1 0 1 (+5)
```

  - As shown above, addition and subtraction of signed-magnitude numbers need corrections.

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Addition and Subtraction

- Two's complement

```
  0 1 0 1 1 (11)       0 1 0 1 1 (11)        1 1 1 0 1 (-3)       1 1 0 0 0 (-8)
+ 0 0 0 1 0 (2)      + 0 0 1 1 0 (6)       + 1 1 0 1 1 (-5)     + 1 0 1 1 1 (-9)
= 0 1 1 0 1 (13)     = 1 0 0 0 1 (-15)     =1 1 1 0 0 0 (-8)    =1 0 1 1 1 1 (-17)
```



```
  0 1 1 0 1 (13)       1 0 0 1 1 (-13)
+ 1 1 0 0 0 (-8)     + 0 1 0 0 0 (8)
= 1 0 0 1 0 1 (5)    = 1 1 0 1 1 (-5)
```

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Arithmetic Shift Operations

- For given $\{x_{n-1}, x_{n-2}, \dots, x_1, x_0\}$
- Signed-magnitude

$$x_{n-1}, 0, 0, \dots, 0, \{x_{n-2}, \dots, x_1, x_0\}, 0, 0, \dots$$

- Two's complement

$$x_{n-1}, x_{n-1}, \dots, x_{n-1}, \{x_{n-1}, x_{n-2}, \dots, x_1, x_0\}, 0, 0, \dots$$

- One's complement

$$x_{n-1}, x_{n-1}, \dots, x_{n-1}, \{x_{n-1}, x_{n-2}, \dots, x_1, x_0\}, x_{n-1}, x_{n-1}, \dots$$

Supplementary Materials

Arithmetic Shift Operations

- Derivation of the extension of the two's complement

$$X = X_{n-1\ldots0} = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$$

If X>0, it is the same as the signed-magnitude case.

If X<0 (if $X \neq -2^n$),

$$X = X_{n-1\ldots0} = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$$

$$-X = \{\overline{X_{n-1\ldots0}} + ulp_0\}$$

$$(-X)_{\text{ext}} = 00\ldots0\{\overline{X_{n-1\ldots0}} + ulp_0\}00\ldots0 = 00\ldots0\{K\}00\ldots0$$

$$-(-X)_{\text{ext}} = \overline{00\ldots0\{K\}00\ldots0} + ulp_{-m} = 11\ldots1\{\overline{K}\}11\ldots1 + ulp_{-m}$$

$$= 11\ldots1\{\overline{K} + ulp_0\}00\ldots0 = 11\ldots1\{2^n - K\}00\ldots0$$

$$= 11\ldots1\{2^n - (\overline{X_{n-1\ldots0}} + ulp_0)\}00\ldots0 = 11\ldots1\{X_{n-1\ldots0}\}00\ldots0$$

$$\therefore\ X_{\text{ext}} = x_{n-1}\ldots x_{n-1}\{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}00\ldots0$$

If $X=2^n$, we can directly prove it.

Arithmetic Shift Operations

- Derivation of the extension of the one's complement

$$X = X_{n-1\ldots0} = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$$

If X>0, it is the same as the signed-magnitude case.

If X<0,

$$X = X_{n-1\ldots0} = \{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}$$

$$-X = \{\overline{X_{n-1\ldots0}}\}$$

$$(-X)_{\text{ext}} = 00\ldots0\{\overline{X_{n-1\ldots0}}\}00\ldots0$$

$$-(-X)_{\text{ext}} = \overline{00\ldots0\{\overline{X_{n-1\ldots0}}\}00\ldots0} = 11\ldots1\{X_{n-1\ldots0}\}11\ldots1$$

$$\therefore\ X_{\text{ext}} = x_{n-1}\ldots x_{n-1}\{x_{n-1}, x_{n-2}, \ldots, x_1, x_0\}x_{n-1}\ldots x_{n-1}$$