- Going forward
- Gridworld
- Q1: Not in-place

Utilities of Sequences

- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- Theorem: only two ways to define stationary utilities
  - Additive utility:

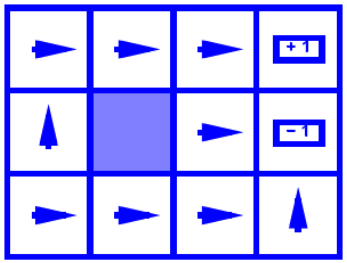
$$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$

  - Discounted utility:

$$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

Infinite Utilities?!

- Problem: infinite state sequences have infinite rewards



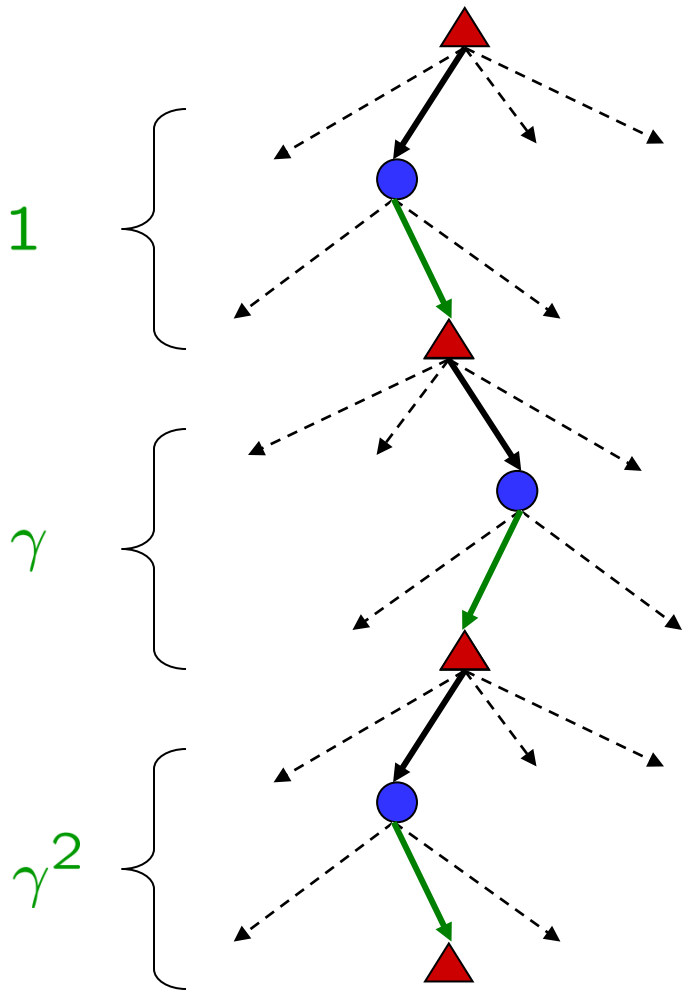
- Solutions:
  - Finite horizon:
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “done” for High-Low)
  - Discounting: for $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller “horizon” – shorter term focus

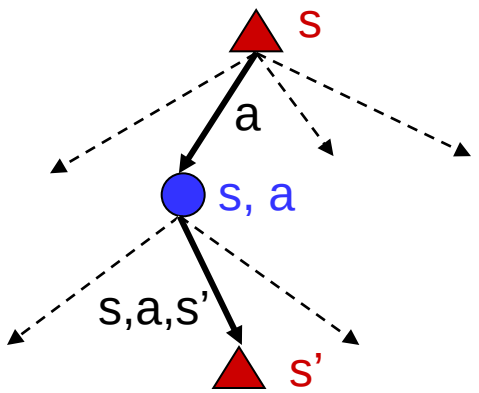
Discounting

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge



Recap: Defining MDPs

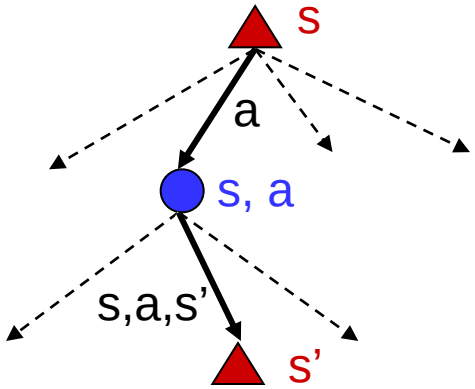
- Markov decision processes:
  - States S
  - Start state s_0
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)

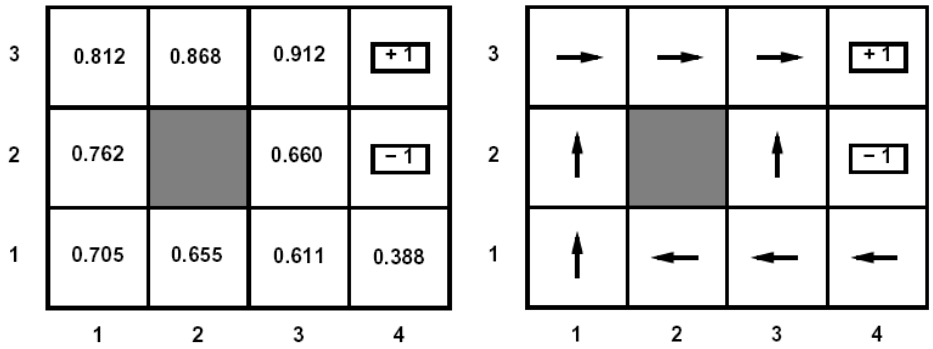


- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility (or return) = sum of discounted rewards

Optimal Utilities

- Fundamental operation: compute the values (optimal expectimax utilities) of states s
- Why? Optimal values define optimal policies!
- Define the value of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- Define the value of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting in s, taking action a and thereafter acting optimally
- Define the optimal policy:
  $\pi^*(s)$ = optimal action from state s

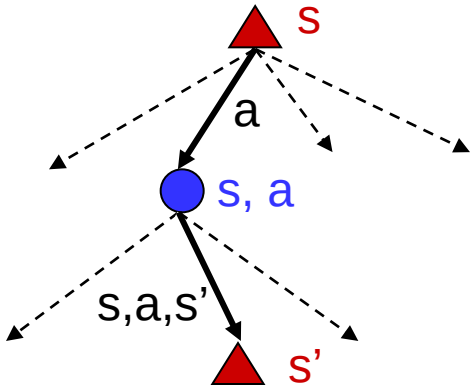




The Bellman Equations

- Definition of "optimal utility" leads to a simple one-step lookahead relationship amongst optimal utility values:

  Optimal rewards = maximize over first action and then follow optimal policy



- Formally:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

Solving MDPs

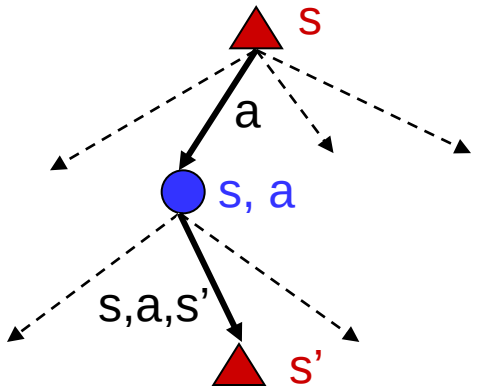
- We want to find the optimal policy π^*

- Proposal 1: modified expectimax search, starting from each state s:

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

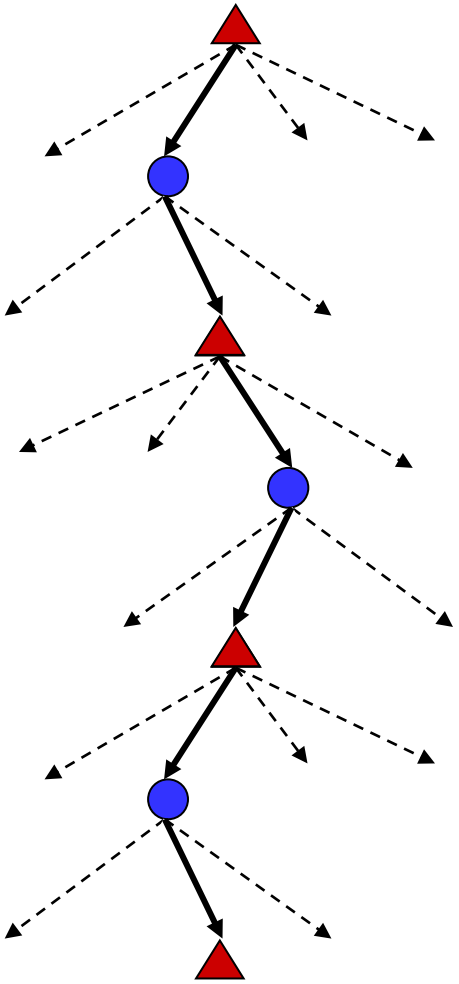
$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a Q^*(s,a)$$



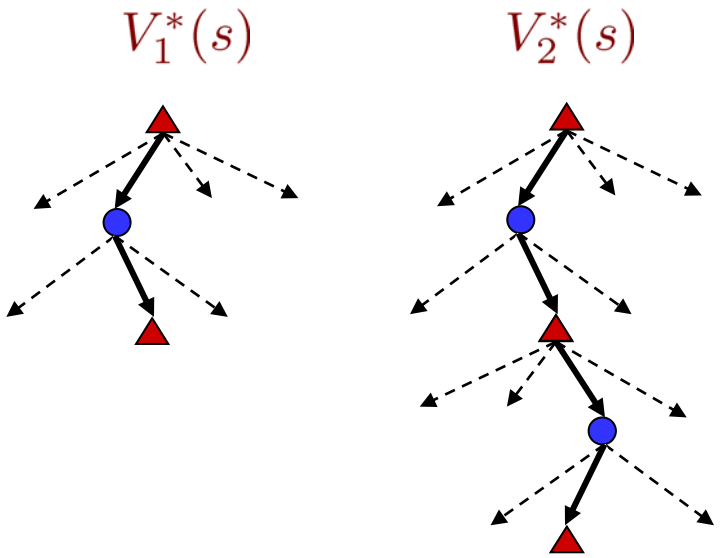
Why Not Search Trees?

- Why not solve with expectimax?
- Problems:
  - This tree is usually infinite (why?)
  - Same states appear over and over (why?)
  - We would search once per state (why?)
- Idea: Value iteration
  - Compute optimal values for all states all at once using successive approximations
  - Will be a bottom-up dynamic program similar in cost to memoization
  - Do all planning offline, no replanning needed!



Value Estimates

- Calculate estimates $V_k^*(s)$
  - Not the optimal value of s!
  - The optimal value considering only next k time steps (k rewards)
  - As k $\rightarrow \infty$, it approaches the optimal value



- Almost solution: recursion (i.e. expectimax)
- Correct solution: dynamic programming

Value Iteration

- Idea:
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for depth i+1:

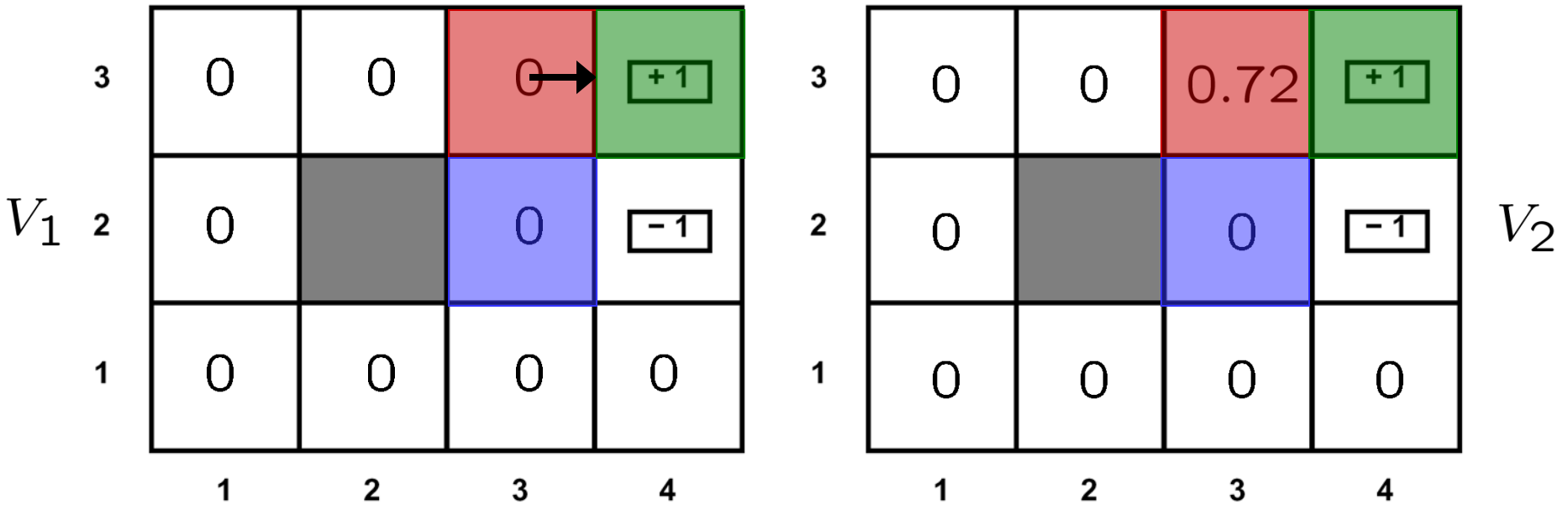
$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s') \right]$$

  - This is called a value update or Bellman update
  - Repeat until convergence

- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Example: γ=0.9, living reward=0, noise=0.2

Example: Bellman Updates



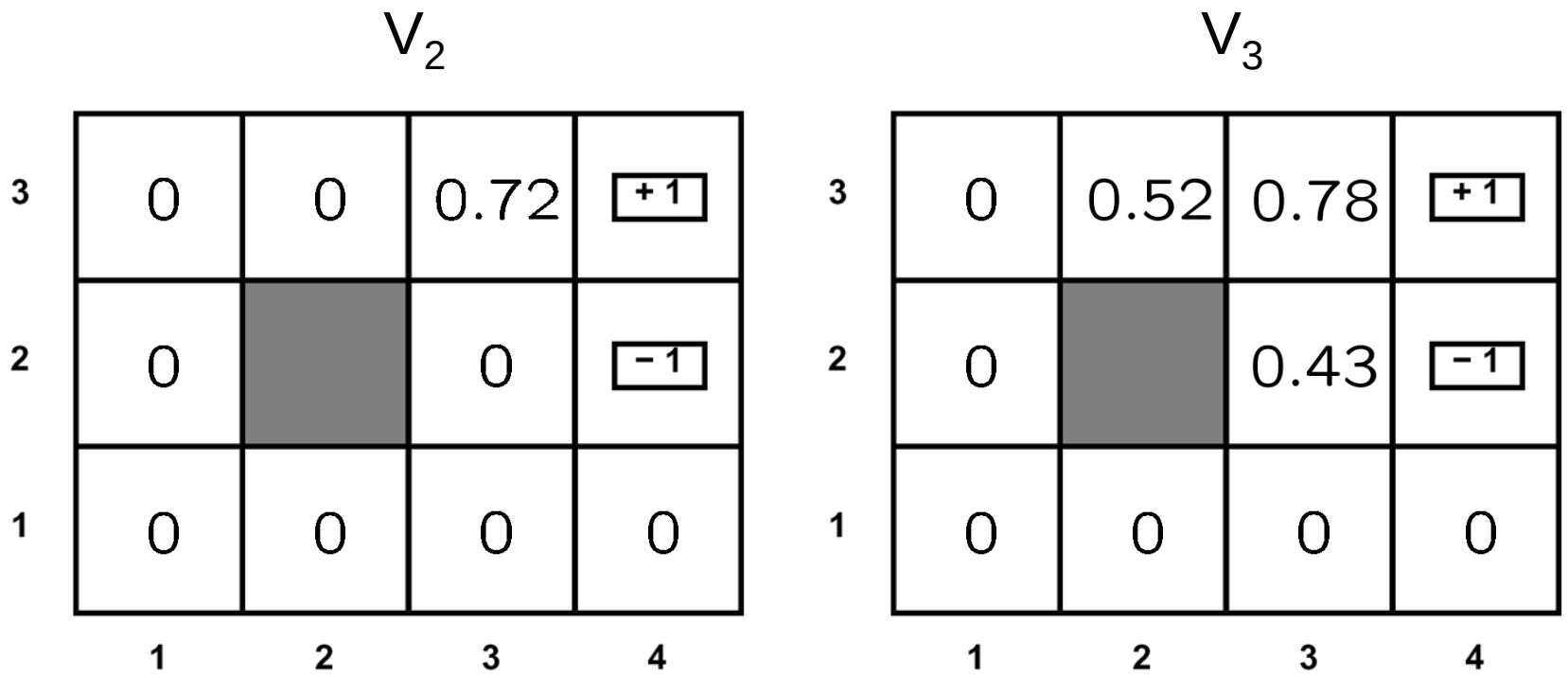
$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s') \right]$$

$$V_2(\langle 3,3 \rangle) = \sum_{s'} T(\langle 3,3 \rangle, \text{right}, s') \left[R(\langle 3,3 \rangle) + 0.9 V_1(s') \right]$$

max happens for a=right, other actions not shown

$$= 0.9 \left[0.8 \cdot 1 + 0.1 \cdot 0 + 0.1 \cdot 0 \right]$$

Example: Value Iteration



- Information propagates outward from terminal states and eventually all states have correct value estimates

Convergence*

- Define the max-norm: $||U|| = \max_s |U(s)|$
- Theorem: For any two approximations U and V

$$||U^{t+1} - V^{t+1}|| \leq \gamma\, ||U^t - V^t||$$

  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution
- Theorem:

$$||U^{t+1} - U^t|| < \epsilon, \Rightarrow ||U^{t+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$

  - I.e. once the change in our approximation is small, it must also be close to correct

Practice: Computing Actions

- Which action should we chose from state s:
  - Given optimal values V?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

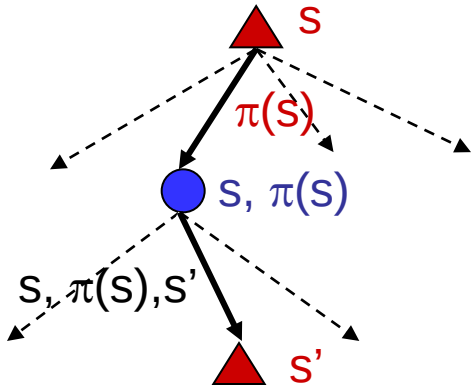
  - Given optimal q-values Q?

$$\arg\max_a Q^*(s,a)$$

  - Lesson: actions are easier to select from Q's!

Utilities for Fixed Policies

- Another basic operation: compute the utility of a state s under a fix (general non-optimal) policy



- Define the utility of a state s, under a fixed policy π:

  $V^{\pi}(s)$ = expected total discounted rewards (return) starting in s and following π

- Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$