Using Feedback to Block Controllability at Remote Nodes in Network Synchronization Processes

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Abstract— **The design of local state-feedback control systems to prevent controllability at remote network nodes is studied. An algorithm based on a joint eigenvalue-right eigenvector assignment method is developed, which under broad conditions maintains all the eigenvalues of the open-loop system while blocking controllability from selected remote nodes. Additionally, the graph structure of the network is exploited to enable controllability-blocking based on regional feedback, where only state measurements in a network partition are required. The design based on regional feedback does not preserve eigenvalue locations, however a modification of the design based on time-scale separation is presented which guarantees stability. The results are illustrated with a numerical example.**

I. Introduction

In the recent years, controllability and observability of dynamical networks has been very widely studied in the controls-engineering community [1]–[12]. This research effort has included the development of graph-theoretic conditions for observability and controllability [2], [7], [9] and the analysis of Gramian-based metrics for control energy and state-estimation fidelity [3], [5], [7] in canonical linear network models (e.g. models for synchronization). In addition, efforts have been made to understand input-output notions of the dynamical networks such as output-controllability and transfer-function zeros [4], [10] and extend the analyses to more sophisticated linear models with hierarchical structures, and some simple classes of nonlinear models [8], [12].

The studies on network controllability and observability are almost exclusively focused on properties of an individual channel (which may be be single input/output or multi input/output). However, network processes today are often managed by multiple distinct authorities with different sensing and actuation capabilities, whose goals may be cooperative, orthogonal or non-cooperative. Thus, there is a need to study not only individual channels in network models, but also to understand how one control action may modulate the properties of other control channels. Motivated by this need, a few groups including ours have begun to study the design of controllers or network structure to shape control properties at other remote network locations [11], [13], [14]. As one effort in this direction, our group studied the design of both state and output feedback controllers in networks, to change the observability property (specifically, prevent observability) at a remote network channel [14]. Here, we study the dual question of controller design to prevent or block controllability at remote node(s) in the network.

The problem considered here – i.e., the impact analysis of control designs on the controllability of remote channels – is relevant to a number of infrastructure-control applications, where multiple control authorities must be coordinated. As one example, bulk power-grid engineers have recognized the potential for malicious control wherein an authority or intruder may destabilize a network mode with relatively little effort, based on selfish motives [15]. The design of feedback to block controllability of a remote channel could be useful in preventing such malicious control. Conversely, the impact of a control design on a remote channel's controllability is also important to ensure that controllers in the grid do not interfere with other authorities' abilities to manage transients. Similar types of control coordination problems arise in e.g. air transportation systems, where flow controls are governed by multiple authorities with distinct aims. Similarly for secured operation of autonomous multi-vehicle system operator/authority may want to limit controllability of intruders who can probe a subset of vehicles [8]. Controllability blocking controller may also be needed, for instance, for Internet-of-Things networks, to regulate the potential harm caused by a cyber-attacker [16].

In this work, we study the design of state feedback controllers applied at a set of network nodes to enforce uncontrollability for the actuation at a different set of nodes, in the context of a standard linear model for network synchronization. A general design algorithm of such *controllability-blocking controllers* based on a joint eigenvalue-right eigenvector assignment technique [17] is developed, which works under broad conditions and preserves all the eigenvalues and a subset of right eigenvectors of open-loop model via full state feedback. Further by exploiting the topological structure of the network we demonstrate that controllability can be blocked using *regional state feedback* scheme which only uses state measurements in a partition of the network. These designs modify the eigenvalues; however we show that stability can still be ensured through time-scale separation based design. As a whole, we note that the designs achieved for controllability-blocking mirror those developed for observability-blocking in our previous work [14], however the design approaches and characterizations differ significantly for the two problems. Relative to [14] the design of controllability-blocking controller presented here is rather more intricate as it involves indirect assignment of left eigenvectors having specific zero pattern.

This work was partially supported by NSF Grants 1545104 and 1635184.

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The organization of the paper as follows. Section II and Section III respectively formulates the controllability-blocking controller design problem and briefly reviews the joint eigenvalue-right eigenvector assignment technique. The main results on controllabilty-blocking controls are presented in Section IV. In Section V we provide a numerical example explaining the proposed algorithm, and in Section VI we make concluding remarks.

II. Problem Formulation

We consider a standard model for network synchronization, which arises in many settings (e.g. in the power grid, robotic teams, and water-distribution systems) [18]–[20]. The synchronization model is augmented to represent actuation at a set of nodes where feedback controls can be applied by a system operator. Our main objective is to design feedback controllers at those actuation nodes such that the dynamics becomes uncontrollable for the actuation at a different set of nodes.

Formally, we define the synchronization model on a weighted digraph $\mathcal{G}(\mathcal{V},\mathcal{E}:\mathcal{W})$. Here, $\mathcal{V}$ denotes the vertex set containing n vertices labeled as $1,2,\ldots,n$. The ordered pair $(i,j)\in\mathcal{E}$ denotes an edge from vertex i to vertex j and corresponding weight is denoted by $w_{ij}\in\mathcal{W}$ which is assumed to be positive. The synchronization dynamics is specified by the (asymmetric) Laplacian or diffusion matrix $\mathbf{L}$ of the graph. Specifically, the entries of $\mathbf{L}\in\mathbb{R}^{n\times n}$ are as follows: each off-diagonal entry L_{ij} is equal to $-w_{ji}$ for all $(i,j)\in\mathcal{E}$, otherwise 0; each diagonal entry L_{ii} is equal to $-\sum_{j=1,j\neq i}^{n}L_{ij}$.

The synchronization model is defined on a network with n nodes labeled $1,\ldots,n$, which correspond to the graph vertices. Each node i associates a scalar state $x_i(t)$ where $\mathbf{x}(t)=[x_1(t)\quad x_2(t)\quad\cdots\quad x_n(t)]^T$ denotes the network state. Actuation can be provided at a set of q ($2\leq q\leq n$) nodes $\{r_1,r_2,\ldots,r_q\}$, which we call *actuation nodes* and denote the set as $\mathcal{R}$ (i.e. $\mathcal{R}=\{r_1,r_2,\ldots,r_q\}$). The model dynamics are then given by:

$$\dot{\mathbf{x}}=-\mathbf{L}\mathbf{x}+\mathbf{B}\mathbf{u} \tag{1}$$

where $\mathbf{u}$ is a q-element vector containing the input signals at the actuation nodes, $\mathbf{B}=[\mathbf{e}_{r_1}\ \mathbf{e}_{r_2}\cdots\ \mathbf{e}_{r_q}]$ and $\mathbf{e}_i$ is a 0-1 indicator vector in $\mathbb{R}^n$ with ith entry equal to 1. Here in this study we assume that the network graph is strongly connected. Thus the network state achieves synchronization for the zero-input response in the sense that the manifold where all the nodes' states are equal is asymptotically stable in the sense of Lyapunov [18].

In this study, we consider the design of linear feedback controllers at the actuation nodes, to modulate the controllability of the network dynamics for the actuation at a different set of m ($1\leq m<n$) nodes given by $\{s_1,s_2,\ldots,s_m\}$. We call them as *target nodes* and denote the set as $\mathcal{S}$ (i.e. $\mathcal{S}=\{s_1,s_2,\ldots,s_m\}$). We stress that the actuation and target nodes may overlap. The input matrix for the actuation at these target nodes is given as $\hat{\mathbf{B}}=[\mathbf{e}_{s_1}\ \mathbf{e}_{s_2}\cdots\ \mathbf{e}_{s_m}]$. Nominally we consider state feedback control scheme. Hence the controller at each node r_i, $i=1,\ldots,q$, is specified as $u_{r_i}=-\mathbf{k}_{r_i}^T\mathbf{x}$, where $\mathbf{k}_{r_i}$ is the control gain. Assembling the state feedback models for each actuation node, we obtain $\mathbf{u}=-\mathbf{F}\mathbf{x}$ where $\mathbf{F}=[\mathbf{k}_{r_1}\ \mathbf{k}_{r_2}\cdots\mathbf{k}_{r_q}]^T$. Upon application of the feedback control, the closed-loop dynamics become:

$$\dot{\mathbf{x}}=-(\mathbf{L}+\mathbf{B}\mathbf{F})\ \mathbf{x} \tag{2}$$

Our initial goal here is to pursue design of the state-feedback controller $\mathbf{F}$ such that the closed-loop model $(-(\mathbf{L}+\mathbf{B}\mathbf{F}),\hat{\mathbf{B}})$ becomes uncontrollable while maintaining as much of the open-loop eigen-structure as possible. Secondly, we study whether regional feedback controls, where only states from a network partition containing the actuators are used in feedback, can be used to block controllability at the target nodes ensuring stability of the closed-loop system.

III. Preliminaries

The proposed algorithm for designing controllability-blocking controllers is based on a method for joint eigenvalue-right eigenvector assignment via linear state feedback control [17], which we briefly review here. The article [17] provides a means to not only place the eigenvalues of a linear system through state feedback, but also to place corresponding right eigenvectors in certain permissible vector spaces. To formalize this concept, we consider the closed-loop system $\dot{\underline{\mathbf{x}}}=(\underline{\mathbf{A}}+\underline{\mathbf{B}}\ \underline{\mathbf{F}})\underline{\mathbf{x}}$ where $\underline{\mathbf{x}}\in\mathbb{R}^n$, $\underline{\mathbf{A}}\in\mathbb{R}^{n\times n}$, $\underline{\mathbf{B}}\in\mathbb{R}^{n\times q}$ and $\underline{\mathbf{F}}\in\mathbb{R}^{q\times n}$ denote state, state matrix, input matrix and feedback gain matrix respectively. Then, for any given $\underline{\lambda}$, we construct a matrix $\underline{\mathbf{N}}(\underline{\lambda})=[\underline{\mathbf{N}}_1(\underline{\lambda})^T\ \ \underline{\mathbf{N}}_2(\underline{\lambda})^T]^T$, where $\underline{\mathbf{N}}(\underline{\lambda})\in\mathbb{C}^{(n+q)\times l},\underline{\mathbf{N}}_1(\underline{\lambda})\in\mathbb{C}^{n\times l},\underline{\mathbf{N}}_2(\underline{\lambda})\in\mathbb{C}^{q\times l}$, $l\geq q$, has the following properties: the columns of $\underline{\mathbf{N}}(\underline{\lambda})$ are linearly independent and span the null-space of $\underline{\mathbf{S}}(\underline{\lambda})=[(\underline{\mathbf{A}}-\underline{\lambda}\ \mathbf{I}_n)\ \ \underline{\mathbf{B}}]$. So,

$$[(\underline{\mathbf{A}}-\underline{\lambda}\ \mathbf{I}_n)\ \ \underline{\mathbf{B}}]\begin{bmatrix}\underline{\mathbf{N}}_1(\underline{\lambda})\\ \underline{\mathbf{N}}_2(\underline{\lambda})\end{bmatrix}=\mathbf{0} \tag{3}$$

Following the above notations the Proposition 1 of [17] can be restated as:

Proposition 1: *Consider the linear time invariant system described above. Assume $\{\underline{\lambda}_1,\ldots,\underline{\lambda}_n\}$ are a self conjugate set of distinct complex numbers. For a given set of complex vectors $\{\underline{\mathbf{v}}_1,\ldots,\underline{\mathbf{v}}_n\}$, there exists a matrix $\underline{\mathbf{F}}$ of real numbers such that $\underline{\lambda}_i\underline{\mathbf{v}}_i=(\underline{\mathbf{A}}+\underline{\mathbf{B}}\ \underline{\mathbf{F}})\underline{\mathbf{v}}_i\ \forall i\in\{1,\ldots,n\}$ if and only if the following three conditions are satisfied $\forall i\in\{1,\ldots,n\}$.*

1) *The vectors $\underline{\mathbf{v}}_1,\ldots,\underline{\mathbf{v}}_n$ form a linearly independent set in $\mathbb{C}^n$.*
2) *$\underline{\mathbf{v}}_i=\underline{\mathbf{v}}_j^*$ whenever $\underline{\lambda}_i=\underline{\lambda}_j^*$*
3) *$\underline{\mathbf{v}}_i$ in the column-space of $\underline{\mathbf{N}}_1(\underline{\lambda}_i)$*

Furthermore, if $\underline{\mathbf{F}}$ exists and rank$(\underline{\mathbf{B}})=q$, then $\underline{\mathbf{F}}$ is unique.

IV. Main Results

We pursue the design of controllability-blocking controllers in two steps. First, we develop a general method for blocking controllability via state feedback control. Then, we pursue the design of regional feedback controllers which uses state measurements in a network partition only.

A. General Design of Controllability-Blocking Controllers

Here we present an algorithm for constructing a state-feedback controller at q ($2 \leq q \leq n$) actuation nodes such that the pair $(-(\mathbf{L}+\mathbf{BF}), \hat{\mathbf{B}})$ is uncontrollable based on the eigenvalue-right eigenvector placement technique reviewed above. Specifically, the algorithm assigns $m+1$ right eigenvectors of the closed-loop state matrix to achieve a desired closed-loop left eigenvector that imposes uncontrollability for the actuation at target nodes. Recall the Popov-Belevitch-Hautus (PBH) test according to which we know that the pair $(-(\mathbf{L}+\mathbf{BF}), \hat{\mathbf{B}})$ is uncontrollable if and only if $(\mathbf{L}+\mathbf{BF})$ has a left eigenvector $\mathbf{w}$ such that $\mathbf{w}^T\hat{\mathbf{B}} = \mathbf{0}$ [22]. Without loss of generality, we assume that the last m nodes (nodes $n-m+1, \ldots, n$) are the target nodes. Hence, $\hat{\mathbf{B}} = [\mathbf{e}_{n-m+1} \cdots \mathbf{e}_n]$ and therefore the pair $(-(\mathbf{L}+\mathbf{BF}), \hat{\mathbf{B}})$ is uncontrollable if only if $(\mathbf{L}+\mathbf{BF})$ has a left eigenvector whose final m entries are zero.

We consider the case that $\mathbf{L}$ is diagonalizable and has n real eigenvalues for our initial development of the algorithm (we comment on more general cases in the remarks). We label the eigenvalues of $\mathbf{L}$ as $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding right eigenvectors are labeled as $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We also define a set V which contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Note V is a set of n linearly independent vectors in $\mathbb{R}^n$. Now we choose one of the eigenvalues of $\mathbf{L}$, say λ_p where $p \in \{1, \ldots, n\}$. Here our aim is to design a feedback control such that the open-loop left eigenvector $\mathbf{w}_p$ associated with λ_p is modified to a vector $\hat{\mathbf{w}}_p$ in closed-loop whose entries corresponding to target nodes are all zeros. To achieve this we will modify $m+1$ right eigenvectors but maintain the remaining right eigenvectors and all the eigenvalues. In this way, we enforce uncontrollability on the mode $(-\lambda_p)$ in the closed-loop system. The design can be achieved by following the steps described below:

Algorithm:

1) First, we choose an eigenvalue λ_p of $\mathbf{L}$ and its associated right eigenvector $\mathbf{v}_p$ where $p \in \{1, 2, \ldots, n\}$.
2) Next we construct the open-loop modal matrix $\mathbf{V}_0 \in \mathbb{R}^{n\times n}$ where $\mathbf{V}_0 = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$. We discard the last m rows (i.e. the rows associated to the target nodes) of $\mathbf{V}_0$ to obtain $\mathbf{V}_1 \in \mathbb{R}^{(n-m)\times n}$.
3) Now we discard $m+1$ of the columns of $\mathbf{V}_1$ including p-th column (i.e. the column associated to $\mathbf{v}_p$) to obtain a matrix $\mathbf{V}_2 \in \mathbb{R}^{(n-m)\times(n-m-1)}$. Suppose the set of column numbers that we choose to discard is $C = \{c_1, c_2, \ldots, c_{m+1}\}$. Note, $c_i \in \mathbb{N}$, $1 \leq c_i \leq n$ and there is $c_i \in C$ such that $c_i = p$.
4) Next we find the nonzero vector $\mathbf{w}_{p1} \in \mathbb{R}^{n-m}$ such that $\mathbf{w}_{p1}^T\mathbf{V}_2 = \mathbf{0}$. We then insert m zeros in the end of $\mathbf{w}_{p1}$ and obtain $\hat{\mathbf{w}}_p$ (i.e. $\hat{\mathbf{w}}_p^T = [\mathbf{w}_{p1}^T\ \ \mathbf{0}]$).
5) For $c_i = p$, we choose a nonzero vector $\hat{\mathbf{v}}_p$ in the column-space of $\mathbf{N}_1(\lambda_p)$ such that $\hat{\mathbf{w}}_p^T\hat{\mathbf{v}}_p \neq 0$. We then set $\mathbf{z}_p = -(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{S}(\lambda_p)\hat{\mathbf{v}}_p$. [1] But if $\hat{\mathbf{w}}_p^T\mathbf{v}_p \neq 0$, we set $\hat{\mathbf{v}}_p = \mathbf{v}_p$ and $\mathbf{z}_p = \mathbf{0}$.
6) Next for each $c_i \in C$ and $c_i \neq p$ we do the following:
   a) If $\hat{\mathbf{w}}_p^T\mathbf{v}_{c_i} = 0$, we set $\hat{\mathbf{v}}_{c_i} = \mathbf{v}_{c_i}$ and $\mathbf{z}_{c_i} = \mathbf{0}$; otherwise we take the below three steps to compute $\hat{\mathbf{v}}_{c_i}$ and $\mathbf{z}_{c_i}$.
   b) We construct the matrix $\mathbf{S}(\lambda_{c_i}) = (\mathbf{L} - \lambda_{c_i}\mathbf{I}_n)$. Then we discard r_i-th rows of $\mathbf{S}(\lambda_{c_i})$ for all $r_i \in \mathcal{R}$ (i.e. the rows associated to the actuation nodes) to obtain $\mathbf{R}(\lambda_{c_i}) \in \mathbb{R}^{(n-q)\times n}$ and then insert $\hat{\mathbf{w}}_p^T$ at the bottom of $\mathbf{R}(\lambda_{c_i})$ to obtain $\mathbf{M}(\lambda_{c_i}) \in \mathbb{R}^{(n-q+1)\times n}$ (i.e. $\mathbf{M}(\lambda_{c_i}) = [\mathbf{R}(\lambda_{c_i})^T\ \ \hat{\mathbf{w}}_p]^T$).
   c) We seek a non-zero vector $\hat{\mathbf{v}}_{c_i}$ in the null-space of $\mathbf{M}(\lambda_{c_i})$. Hence, we find $\hat{\mathbf{v}}_{c_i}$ such that:
   $$\mathbf{M}(\lambda_{c_i})\hat{\mathbf{v}}_{c_i} = \mathbf{0}; \quad \hat{\mathbf{v}}_{c_i} \neq \mathbf{0}. \tag{4}$$
   d) Next we calculate the vector $\mathbf{z}_{c_i}$ which is given by:
   $$\mathbf{z}_{c_i} = -(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{S}(\lambda_{c_i})\hat{\mathbf{v}}_{c_i}. \tag{5}$$
7) We construct the matrix $\mathbf{V} \in \mathbb{R}^{n\times n}$ and $\mathbf{Z} \in \mathbb{R}^{q\times n}$ which are given by:
   $$\mathbf{V} = [\hat{\mathbf{v}}_{c_1} \cdots \hat{\mathbf{v}}_{c_{m+1}}\ \mathbf{V}_3], \tag{6}$$
   $$\mathbf{Z} = [\mathbf{z}_{c_1} \cdots \mathbf{z}_{c_{m+1}}\ \mathbf{0}]. \tag{7}$$
   Here $\mathbf{V}_3 \in \mathbb{R}^{(n-m-1)\times n}$ is the matrix obtained by discarding the c_i-th column of $\mathbf{V}_0$ for all $c_i \in C$.
8) If $\mathbf{V}$ is invertible, then gain matrix of controllability blocking controller $\mathbf{F}$ is given by (8):
   $$\mathbf{F} = \mathbf{Z}\ \mathbf{V}^{-1} \tag{8}$$

□

Now we will justify the above algorithm in designing controllability blocking feedback controller. To do so first we will derive an equivalent statement to the third condition of Proposition 1. Specifically, now we will show that the column-space of $\mathbf{N}_1(\lambda)$ is same as the null-space of $\mathbf{R}(\lambda)$ where the matrix $\mathbf{R}(\lambda) \in \mathbb{C}^{(n-q)\times n}$ is constituted from $\mathbf{S}(\lambda) = (\mathbf{L} - \lambda_i\mathbf{I}_n)$ by omitting the rows corresponding to the actuation nodes (i.e. omitting all r-th rows of $\mathbf{S}(\lambda)$ where $r \in \mathcal{R}$, see Step 5b of the algorithm). We formalize this result as Lemma 1 below.

Lemma 1: *$\hat{\mathbf{v}}$ is in the column-space of $\mathbf{N}_1(\lambda)$ if and only if $\hat{\mathbf{v}}$ is in the null-space of $\mathbf{R}(\lambda)$.*

Proof: According to (3) for any arbitrary vector $\mathbf{k} \in \mathbb{C}^l$ we can write $(\mathbf{L} - \lambda\mathbf{I}_n)\ \mathbf{N}_1(\lambda)\mathbf{k} = -\mathbf{B}\mathbf{N}_2(\lambda)\mathbf{k}$ or equivalently, $(\mathbf{L} - \lambda\mathbf{I}_n)\hat{\mathbf{v}} = \mathbf{Bz}$ where $\hat{\mathbf{v}} = \mathbf{N}_1(\lambda)\mathbf{k}$ and $\mathbf{z} = -\mathbf{N}_2(\lambda)\mathbf{k}$. Now for the convenience of representation without loss of generality assume just for this proof that the first q nodes are actuation nodes. Therefore $\mathcal{S} = \{1, 2, \cdots, q\}$ and $\mathbf{B} = [\mathbf{e}_1\ \mathbf{e}_2\ \ldots\ \mathbf{e}_q]$. So now we can write

$$\begin{bmatrix} \mathbf{R}_0(\lambda)\ \hat{\mathbf{v}} \\ \mathbf{R}(\lambda)\ \hat{\mathbf{v}} \end{bmatrix} = (\mathbf{L} - \lambda\mathbf{I}_n)\hat{\mathbf{v}} = \mathbf{Bz} = \begin{bmatrix} \mathbf{I}_q \\ \mathbf{0} \end{bmatrix}\mathbf{z} = \begin{bmatrix} \mathbf{z} \\ \mathbf{0} \end{bmatrix} \tag{9}$$

where $\mathbf{R}_0(\lambda) \in \mathbb{C}^{q\times n}$ is the matrix constituted from the first q rows of $(\mathbf{L} - \lambda\mathbf{I}_n)$. From (9) it is clear that $\hat{\mathbf{v}}$ is in the column-space of $\mathbf{N}_1(\lambda)$ if and only if $\mathbf{R}(\lambda)\hat{\mathbf{v}} = \mathbf{0}$.

□

[1] When such $\hat{\mathbf{v}}_p$ does not exist, we choose different C in Step 3 or different λ_p in Step 1.

Now using the above lemma in tandem with Moore's results the outcome of the design algorithm can be formalized as:

Theorem 1: *Consider the network synchronization model, and assume there are q ($2 \leq q \leq n$) number of actuation nodes. Assume that: 1) $\mathbf{L}$ has real eigenvalues and n linearly-independent right eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ (i.e. is diagonalizable), and 2) the modified modal matrix $\mathbf{V}$ as obtained in the Step 7 of presented algorithm is invertible. Then the state feedback controller $\mathbf{F}$ obtained from (8) blocks controllability of the model in the target nodes, i.e. the pair $(-(\mathbf{L}+\mathbf{BF}), \hat{\mathbf{B}})$ is uncontrollable. Furthermore, all the eigenvalues and the right eigenvectors of the open-loop model except $\mathbf{v}_{c_1}, \ldots, \mathbf{v}_{c_{m+1}}$ are maintained in the closed-loop model.*

Proof:

We will prove the theorem in two steps. In the first step we will show that all the open-loop eigenvalues and right eigenvectors except $\{\mathbf{v}_{c_1}, \ldots, \mathbf{v}_{c_m}, \mathbf{v}_{c_{m+1}}\}$ are maintained in the closed-loop system. In the second step we will show that $\hat{\mathbf{w}}_p$ is a closed-loop left eigenvector which also satisfies PBH condition of uncontrollability for the actuation at target nodes.

We begin with the first step by noting that from (6), (7) and (8) we get $\mathbf{FV}_3 = \mathbf{0}$. According to the definition of $\mathbf{V}_3$ given in Step 7 it is immediate that $(\lambda_i, \mathbf{v}_i)$ are the eigenvalue and right eigenvector pair of $(\mathbf{L}+\mathbf{BF})$ for all $i \in \{1, 2, \ldots, n\} \backslash C$. Next we will show that $(\lambda_{c_i}, \hat{\mathbf{v}}_{c_i})$ are the eigenvalue and right eigenvector pair of $(\mathbf{L}+\mathbf{BF})$ for all $c_i \in C$. Note, $\mathbf{V}_2 \in \mathbb{R}^{(n-m)\times(n-m-1)}$ and so a nonzero vector $\mathbf{w}_{p1} \in \mathbb{R}^{n-m}$ exists in Step 4 such that $\mathbf{w}_{p1}^T \mathbf{V}_2 = \mathbf{0}$. Now consider the case when $c_i \neq p$. Note for $q \geq 2$, $\mathbf{M}(\lambda_{c_i}) \in \mathbb{R}^{(n-q+1)\times n}$ is rank-deficient and thus a nonzero vector $\hat{\mathbf{v}}_{c_i}$ exists in the null-space of $\mathbf{M}(\lambda_{c_i})$. Since $\mathbf{M}(\lambda_{c_i}) = [\mathbf{R}(\lambda_{c_i})^T \;\; \hat{\mathbf{w}}_p]^T$, so from (4) it is clear that $\hat{\mathbf{v}}_{c_i}$ obtained in Step 6c is in the null-space of $\mathbf{R}(\lambda_{c_i})$. Now using Lemma 1 we can say that $\hat{\mathbf{v}}_{c_i}$ is also in the column-space of $\mathbf{N}_1(\lambda_{c_i})$ and so there exists a vector $\mathbf{k}_{c_i}$ such that $\hat{\mathbf{v}}_{c_i} = \mathbf{N}_1(\lambda_{c_i})\mathbf{k}_{c_i}$. According to (3) solving for $\mathbf{N}_2(\lambda_{c_i})\mathbf{k}_{c_i}$ we obtain $\mathbf{N}_2(\lambda_{c_i})\mathbf{k}_{c_i} = -(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{S}(\lambda_{c_i})\hat{\mathbf{v}}_{c_i}$. Thus from (5) we can write $\mathbf{z}_{c_i} = \mathbf{N}_2(\lambda_{c_i})\mathbf{k}_{c_i}$. On the other hand from (8) we get $\mathbf{F}\hat{\mathbf{v}}_{c_i} = \mathbf{z}_{c_i}$. Now we will use arguments of [17] to prove remaining parts of the first step of proof. According to (3) we can write $(\mathbf{L}-\lambda_{c_i}\mathbf{I}_n)\, \mathbf{N}_1(\lambda_{c_i})\mathbf{K}_{c_i} + \mathbf{BN}_2(\lambda_{c_i})\mathbf{K}_{c_i}$ $= (\mathbf{L}-\lambda_{c_i}\mathbf{I}_n)\, \hat{\mathbf{v}}_{c_i} + \mathbf{Bz}_{c_i} = (\mathbf{L}-\lambda_{c_i}\mathbf{I}_n)\, \hat{\mathbf{v}}_{c_i} + \mathbf{BF}\hat{\mathbf{v}}_{c_i} = \mathbf{0}$ or equivalently, $(\mathbf{L}+\mathbf{BF})\hat{\mathbf{v}}_{c_i} = \lambda_{c_i}\hat{\mathbf{v}}_{c_i}$. Thus $(\lambda_{c_i}, \hat{\mathbf{v}}_{c_i})$ are the eigenvalue and right eigenvector pair of $(\mathbf{L}+\mathbf{BF})$ for all $c_i \in C$ and $c_i \neq p$. Since $\hat{\mathbf{v}}_p$ in the column-space of $\mathbf{N}_1(\lambda_p)$, using same argument we can show that $(\lambda_p, \hat{\mathbf{v}}_p)$ are the eigenvalue and right eigenvector pair of $(\mathbf{L}+\mathbf{BF})$.

Now in the second step of our proof we will show that $\hat{\mathbf{w}}_p$ is a left eigenvector of $(\mathbf{L}+\mathbf{BF})$ by using biorthogonality between right eigenvectors and left eigenvectors. Since $\mathbf{M}(\lambda_{c_i}) = [\mathbf{R}(\lambda_{c_i})^T \;\; \hat{\mathbf{w}}_p]^T$, so from (4) it is clear that $\hat{\mathbf{w}}_p$ is orthogonal to all the vectors in the set $\{\hat{\mathbf{v}}_{c_1}, \ldots, \hat{\mathbf{v}}_{c_m}, \hat{\mathbf{v}}_{c_{m+1}}\}$ except $\hat{\mathbf{v}}_p$. Since $\hat{\mathbf{w}}_p^T = [\mathbf{w}_{p1}^T \;\; \mathbf{0}]$ and $\mathbf{w}_{p1}^T\mathbf{V}_2 = \mathbf{0}$, therefore $\mathbf{w}_p$ is also orthogonal to all the vectors in the set $V \backslash \{\mathbf{v}_{c_1}, \ldots, \mathbf{v}_{c_m}, \mathbf{v}_{c_{m+1}}\}$ according to our construction of $\mathbf{V}_2$ (Recall the Steps 2, 3 and 4 of our algorithm). Also, from construction $\hat{\mathbf{w}}_p^T\hat{\mathbf{v}}_p \neq 0$. Since $\hat{\mathbf{w}}_p$ is orthogonal to all the right eigenvectors of $(\mathbf{L}+\mathbf{BF})$ except $\hat{\mathbf{v}}_p$, therefore it is immediate that $\hat{\mathbf{w}}_p$ is the left eigenvector of $(\mathbf{L}+\mathbf{BF})$ corresponding to the eigenvalue at λ_p. As the final m entries of $\hat{\mathbf{w}}_p$ is zero, hence the pair $(-(\mathbf{L}+\mathbf{BF}), \hat{\mathbf{B}})$ is uncontrollable. □

It should be noted that in the presented algorithm we need to check and ensure that $\mathbf{V}$ is invertible in Step 7. If $\mathbf{V}$ is not invertible, then we cannot obtain $\mathbf{F}$ from (8) and, so the algorithm will not work. Note, $\mathbf{V}$ is invertible only if the set $\hat{V} = V \backslash \{\mathbf{v}_{c_1}, \ldots, \mathbf{v}_{c_{m+1}}\} \cup \{\hat{\mathbf{v}}_{c_1}, \ldots, \hat{\mathbf{v}}_{c_{m+1}}\}$ has n linearly independent vectors. Therefore in Step 5 and 6c we have to select $\{\hat{\mathbf{v}}_{c_1}, \ldots, \hat{\mathbf{v}}_{c_{m+1}}\}$ carefully such that $\hat{V}$ is a set of n linearly independent vectors. If we are not able to find such $\{\hat{\mathbf{v}}_{c_1}, \ldots, \hat{\mathbf{v}}_{c_{m+1}}\}$, then we can either (i) choose a different C in Step 3 or, (ii) choose a different λ_p in Step 1. If still we cannot find any desired $\{\hat{\mathbf{v}}_{c_1}, \ldots, \hat{\mathbf{v}}_{c_{m+1}}\}$ for all such choices, we may conclude that there does not exist any state feedback that can block controllability in the target nodes maintaining all the open-loop eignvalues and selected subset of open-loop right eigenvectors. In fact for some specific sets of actuation and target nodes controllability blocking controller does not exist and the presented algorithm will surely fail. The following lemma provides an instance of such cases.

Lemma 2: *If the pair $(\mathbf{L}, \mathbf{B})$ is controllable and the target nodes contain the actuation nodes (i.e. $\mathcal{R} \subset \mathcal{S}$), no state-feedback controller can exist that blocks controllability of the model in the target nodes.*

The Lemma 2 directly follows from the fact that the rank of controllability matrix is invariant to state feedback and hence we have omitted the proof. This lemma implies that there are certain cases for which our design algorithm will not work for any choices of λ_p and C. In such cases we will have to suitably select a different set of actuation nodes to obtain desired controllability blocking controller per our algorithm. We conjecture that our algorithm can achieve the desired controller when $q = m+1$ actuation nodes are used. We hope to formalize this in our future study.

Several further remarks on the controllability blocking controller design algorithm are worthwhile:

1) The design algorithm presented does not depend on the Laplacian structure of the dynamics. In Section IV.B we utilize the graph topology of the network to obtain regional feedback controls.
2) For undirected graph the condition 1 of Theorem 1 is automatically satisfied since $\mathbf{L}$ is symmetric.
3) Recall in Step 6a (and Step 5) when $\hat{\mathbf{w}}_p^T\mathbf{v}_{c_i} = 0$ (and $\hat{\mathbf{w}}_p^T\mathbf{v}_{c_i} \neq 0$), we choose $\hat{\mathbf{v}}_{c_i} = \mathbf{v}_{c_i}$. When this happens, we maintain open-loop right eigenvector $\mathbf{v}_{c_i}$ in the closed-loop system in addition to those specified in Theorem 1.
4) The algorithm may also be applied when $\mathbf{L}$ has complex eigenvalues with some modifications. The

main idea is that when an eigenvector is modified its conjugate eigenvector has to be modified accordingly such that they remain as a conjugate pair after modifications. We restrict our discussion to real eigenvalues for simplicity (see [17] for additional details).

5) We stress that the controller obtained using our algorithm maintains all the open-loop eigenvalues and $(n-m-1)$ right eigenvectors. Maintenance of eigenvalues and a subset of right eigenvectors in the design is useful for many reasons: i) the design importantly maintains the stability of the open-loop system and, ii) sequential design of feedback controllers for additional goals beyond enforcing uncontrollability is possible. As instance, we can first apply controllers to achieve a performance goal (e.g. eigenvalue placement) or to enforce observability to certain channels, and then a second control loop can be applied to block controllability using our algorithm without modifying the assigned eigenvalues or unobservable right eigenvector.
6) For a controllable system the diagonalizability condition in Theorem 1 can be met since in a controllable system the eigenvalues of the closed-loop system can be placed anywhere in s-plane by using state feedback controller. There are several methods e.g. [23], [24], [25] to assign eigenvalues in a multi-input controllable system which can be applied to make the eigenvalues distinct prior to the design of controllability blocking controller.

B. Controllability-Blocking Using Regional State feedback Controllers

Sometimes adversarial control authority in the network does not have access to the full network state but still seeks to block controllability at remote nodes. To address these situations in this section we consider the case where control authorities seek to block controllability at the target nodes using only the measurement of states in a region or partition of the network. Here we demonstrate that controllability can still be blocked in such case by blocking controllability on nodes associated to vertex-cutset which separates the actuation and target vertices (i.e. vertices associated to the actuation and target nodes). We present a general design of such regional state feedback controller in Theorem 2 which however does not necessarily maintain stability. Then in Theorem 3 we present a design based on time-scale separating control which can maintain stability.

For formalism we begin with defining two synchronization network models. Let us first consider an arbitrary synchronization network model with actuation and target nodes, as defined in Section II; we refer to this model as the *base synchronization network model*, and the state and input matrices for the model are denoted as $\mathbf{L}$ and $\mathbf{B}$ respectively. We also denote input matrix for actuation in the target nodes as $\hat{\mathbf{B}}$. Next, we consider a vertex-cutset $\mathcal{V}_{cut}$ on the network graph, which separates the actuation and target vertices in the base model, as illustrated in Fig. 1. Therefore removal of the vertex cutset $\mathcal{V}_{cut}$ results two disconnected vertex-sets

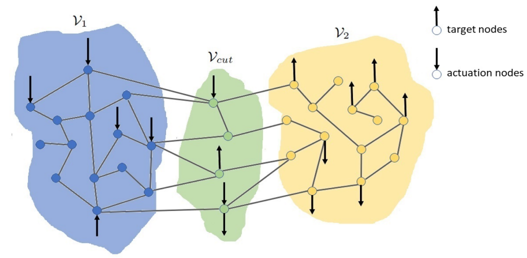

Fig. 1. Graph $\mathcal{G}$ of the network with vertex-cutset separating actuation and target nodes.

$\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1$ does not include any target vertices and $\mathcal{V}_2$ does not include any actuation vertices. Note that the cutset may itself include actuation and/or target vertices (as shown in the figure). Our aim is to design controllability-blocking controller which does not use the state of any node associated to a vertex in $\mathcal{V}_2$.

Now let us define $\tilde{\mathcal{G}}$ to be the subgraph of $\mathcal{G}$ induced by the vertex-set $\mathcal{V}_1 \cup \mathcal{V}_{cut}$. We denote the Laplacian matrix associated with this subgraph $\tilde{\mathcal{G}}$ as $\tilde{\mathbf{L}}$. Now we consider that this smaller network defined on $\tilde{\mathcal{G}}$ has the same actuation nodes as the base synchronization network model but the target nodes are different and defined as the nodes associated to vertex-cutset $\mathcal{V}_{cut}$. We refer to this model as *regional synchronization network model*, and denote the input matrices for actuation and target nodes as $\tilde{\mathbf{B}}$ and $\tilde{\hat{\mathbf{B}}}$. For convenience, we call the network defined on the subgraph of $\mathcal{G}$ induced by the vertex set $\mathcal{V}_2$ as the *residual synchronization network* and denote the associated Laplacian matrix as $\tilde{\mathbf{L}}_{res}$. Without loss of generality we re-number the nodes and associated vertices in this manner: we assign numbers to the nodes associated to $\mathcal{V}_1$ first, then to the nodes associated to $\mathcal{V}_{cut}$ and $\mathcal{V}_2$ respectively. Now we present our initial result on controllability-blocking control using regional state feedback, in the following theorem.

Theorem 2: *Consider a base synchronization network model and associated regional synchronization network model. A state-feedback controller* $\mathbf{F}$ *can be designed to block controllability in the base synchronization model (i.e. to make the pair* $(-(\mathbf{L}+\mathbf{BF}), \hat{\mathbf{B}})$ *uncontrollable) with appropriate zero padding in the controller* $\tilde{\mathbf{F}}$ $\big($ *i.e.* $\mathbf{F} = [\tilde{\mathbf{F}} \;\; \mathbf{0}]$ $\big)$ *which is designed to block controllability in the regional synchronization network model (i.e. to make the pair* $(-(\tilde{\mathbf{L}}+\tilde{\mathbf{B}}\tilde{\mathbf{F}}), \tilde{\hat{\mathbf{B}}})$ *uncontrollable).*

Proof:

Consider a synchronization network defined on graph $\mathcal{G} = (\mathcal{V}, \mathcal{E} : \mathcal{W})$ and vertex-sets $\mathcal{V}_1$, $\mathcal{V}_{cut}$ and $\mathcal{V}_2$ (see Fig. 1). Due to our re-numbering of nodes the Laplacian $\mathbf{L}$ can be partitioned into block as:

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{\mathcal{V}_1\mathcal{V}_1} & \mathbf{L}_{\mathcal{V}_1\mathcal{V}_{cut}} & \mathbf{0} \\ \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_1} & \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} & \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_2} \\ \mathbf{0} & \mathbf{L}_{\mathcal{V}_2\mathcal{V}_{cut}} & \mathbf{L}_{\mathcal{V}_2\mathcal{V}_2} \end{bmatrix}$$

where $\mathbf{L}_{\mathcal{V}_1\mathcal{V}_{cut}}$ refers to the block of $\mathbf{L}$ whose rows and columns correspond to the vertices of $\mathcal{V}_1$ and $\mathcal{V}_{cut}$ respectively (other blocks also refer similarly). Now the Laplacian $\tilde{\mathbf{L}}$ can also be partitioned into block as:

$$\tilde{\mathbf{L}} = \begin{bmatrix} \tilde{\mathbf{L}}_{\mathcal{V}_1\mathcal{V}_1} & \tilde{\mathbf{L}}_{\mathcal{V}_1\mathcal{V}_{cut}} \\ \tilde{\mathbf{L}}_{\mathcal{V}_{cut}\mathcal{V}_1} & \tilde{\mathbf{L}}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{\mathcal{V}_1\mathcal{V}_1} & \mathbf{L}_{\mathcal{V}_1\mathcal{V}_{cut}} \\ \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_1} & \tilde{\mathbf{L}}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} \end{bmatrix}$$

Now consider we apply state feedback controller $\tilde{\mathbf{F}}$ to block controllability for $\hat{\tilde{\mathbf{B}}}$ in the regional synchronization network model. Let $(-\tilde{\lambda}_p)$ and $\tilde{\mathbf{w}}_p = [\tilde{\mathbf{w}}_{p\mathcal{V}_1}^T \;\; \mathbf{0}^T]^T$ are the uncontrollable mode and corresponding left eigenvector. Now we can re-write the left eigenvector equation $\tilde{\mathbf{w}}_p^T(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}) = \tilde{\lambda}_p \tilde{\mathbf{w}}_p^T$ as:

$$\begin{bmatrix} \tilde{\mathbf{w}}_{p\mathcal{V}_1} \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \mathbf{L}_{\mathcal{V}_1\mathcal{V}_1} + \mathbf{F}_{\mathcal{V}_1\mathcal{V}_1} & \mathbf{L}_{\mathcal{V}_1\mathcal{V}_{cut}} + \mathbf{F}_{\mathcal{V}_1\mathcal{V}_{cut}} \\ \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_1} + \mathbf{F}_{\mathcal{V}_{cut}\mathcal{V}_1} & \tilde{\mathbf{L}}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} + \mathbf{F}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} \end{bmatrix} = \tilde{\lambda}_p \begin{bmatrix} \tilde{\mathbf{w}}_{p\mathcal{V}_1} \\ \mathbf{0} \end{bmatrix}^T \quad (10)$$

Here $\mathbf{F}_{\mathcal{V}_1\mathcal{V}_{cut}}$ refers to the actuation on the nodes associated to $\mathcal{V}_1$ exerted by the nodes associated to $\mathcal{V}_{cut}$ through state feedback. Note, $\mathbf{B} = [\tilde{\mathbf{B}}^T \;\; \mathbf{0}^T]^T$. Now let us consider $\mathbf{F} = [\tilde{\mathbf{F}} \;\; \mathbf{0}]$ and $\mathbf{w}_p = [\tilde{\mathbf{w}}_p^T \;\; \mathbf{0}^T]^T$. Then using the block expression of the Laplacians and (10) we can rewrite $\mathbf{w}_p^T(\mathbf{L} + \mathbf{BF})$ as:

$$\begin{bmatrix} \tilde{\mathbf{w}}_{p\mathcal{V}_1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \mathbf{L}_{\mathcal{V}_1\mathcal{V}_1} + \mathbf{F}_{\mathcal{V}_1\mathcal{V}_1} & \mathbf{L}_{\mathcal{V}_1\mathcal{V}_{cut}} + \mathbf{F}_{\mathcal{V}_1\mathcal{V}_{cut}} & \mathbf{0} \\ \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_1} + \mathbf{F}_{\mathcal{V}_{cut}\mathcal{V}_1} & \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} + \mathbf{F}_{\mathcal{V}_{cut}\mathcal{V}_{cut}} & \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_2} \\ \mathbf{0} & \mathbf{L}_{\mathcal{V}_4\mathcal{V}_{cut}} & \mathbf{L}_{\mathcal{V}_2\mathcal{V}_2} \end{bmatrix} = \tilde{\lambda}_p \mathbf{w}_p^T \quad (11)$$

Therefore, $\tilde{\lambda}_p$ and $\mathbf{w}_p$ are the eigenvalue and associated left eigenvector of $(\mathbf{L} + \mathbf{BF})$. Since all the entries of $\mathbf{w}_p$ corresponding to the vertices in $\mathcal{V}_{cut} \cup \mathcal{V}_2$ (which includes all the target vertices) are zero, therefore $(-\tilde{\lambda}_p)$ is an uncontrollable mode for the pair $(-(\mathbf{L} + \mathbf{BF}), \hat{\mathbf{B}})$. $\square$

Theorem 2 is the basis of our design of regional state feedback controllability blocking controller when states of the nodes in regional synchronization network model are accessible by the network control authority. First using our algorithm we try to design controllability blocking controller $\tilde{\mathbf{F}}$ in the regional synchronization network model. When such $\tilde{\mathbf{F}}$ exists, Theorem 2 implies that $\mathbf{F} = [\tilde{\mathbf{F}} \;\; \mathbf{0}]$ blocks controllability in the base synchronization network model. Since the entries of $\mathbf{F}$ associated to the vertices in $\mathcal{V}_2$ are zero, the states of the nodes associated to $\mathcal{V}_2$ are not required to enforce uncontrollability in the base model. Only the states of nodes of regional model are required for blocking controllability in the base model. This result can be useful to many real-world networks whose graphs are known to be sparse e.g. power grids, traffic and communication networks, scale-free networks [26]. Controllability should be blocked at remote nodes of these sparse networks using our regional state feedback design scheme.

Note that Theorem 2 has a serious limitation in designing controllability blocking controller. The designed feedback modifies all the open-loop eigenvalues of the base synchronization network model except $\tilde{\lambda}_p$. We cannot assert in prior that $\mathbf{F}$ obtained by Theorem 2 will maintain stability in the base synchronization network model. Hence after computing $\mathbf{F}$ we need to check the stability of closed-loop system $(-(\mathbf{L} + \mathbf{BF}))$. However when $(\tilde{\mathbf{L}}, \tilde{\mathbf{B}})$ is controllable, the above proposed controller can be modified to guarantee stability using time-scale separation principle. To do so, we first design a controller $\tilde{\mathbf{F}}_1$ to assign all the eigenvalues of $(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}_1)$ such that the eigenvalues are distinct and have real parts greater than d where d is a sufficiently large positive number. Then using our proposed algorithm we try to design the controller $\tilde{\mathbf{F}}_2$ to make the pair $(-(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}), \hat{\tilde{\mathbf{B}}})$ uncontrollable maintaining the assigned eigenvalues where $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_1 + \tilde{\mathbf{F}}_2$. When such $\tilde{\mathbf{F}}_2$ exists, Theorem 2 implies that $\mathbf{F} = [\tilde{\mathbf{F}} \;\mathbf{0}]$ blocks controllability in the base synchronization network model. For sufficiently large d, it can be further shown that the controller $\mathbf{F}$ maintains stability in the closed-loop base model. We formalize this result in the following theorem.

Theorem 3: *Consider a base synchronization network model and associated regional synchronization network model. A state-feedback controller* $\mathbf{F}$ *can be designed to block controllability in the base synchronization model (i.e. to make the pair* $(-(\mathbf{L} + \mathbf{BF}), \hat{\mathbf{B}})$ *uncontrollable) with appropriate zero padding in the controller* $\tilde{\mathbf{F}}$ $\big($ *i.e.* $\mathbf{F} = [\tilde{\mathbf{F}} \;\; \mathbf{0}]$ $\big)$ *which is designed to block controllability in the regional synchronization network model (i.e. to make the pair* $(-(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}), \hat{\tilde{\mathbf{B}}})$ *uncontrollable). Furthermore, when all the eigenvalues of* $(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}})$ *have sufficiently large real parts, the controller* $\mathbf{F}$ *maintains stability in the closed-loop base synchronization network model (i.e.* $-(\mathbf{L} + \mathbf{BF})$ *is stable matrix).*

Proof:

Here we only require to prove the last sentence of Theorem 3 since rest are identical to Theorem 2. Now, (10) and (11) implies that we can write the closed-loop system of the base synchronization network model as

$$(\mathbf{L} + \mathbf{BF}) = \begin{bmatrix} (\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}}) + \mathbf{P}_1 & \mathbf{L}_{reg,res} \\ \mathbf{L}_{res,reg} & \mathbf{L}_{\mathcal{V}_2\mathcal{V}_2} \end{bmatrix} \quad (12)$$

Here, $\mathbf{L}_{reg,res} = [\mathbf{0}^T \;\; \mathbf{L}_{\mathcal{V}_{cut}\mathcal{V}_2}^T]^T$, $\mathbf{L}_{res,reg} = [\mathbf{0} \;\; \mathbf{L}_{\mathcal{V}_2\mathcal{V}_{cut}}]$ and, $\mathbf{P}_1$ is a positive semi-definite diagonal matrix. Note, when the selected value for d is very large, all the eigenvalues of $(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}})$ have very large real parts and thereupon the states of the nodes of regional synchronization network model have very fast dynamics. Then according to the singular perturbation theory [27] we can say that the closed-loop system of the base synchronization network model is stable if real $\lambda\{\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}} + \mathbf{P}_1\} > 0$ and real $\lambda\{\mathbf{L}_{\mathcal{V}_4\mathcal{V}_4} - \mathbf{L}_{res,red}\,(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}} + \mathbf{P}_1)^{-1}\,\mathbf{L}_{red,res}\} > 0$. Sufficiently large d in the design reduces these conditions to real $\lambda\{\mathbf{L}_{\mathcal{V}_4\mathcal{V}_4}\} > 0$ which is true as $\mathbf{L}_{\mathcal{V}_4\mathcal{V}_4}$ is a grounded Laplacian. $\square$

V. Numerical Example

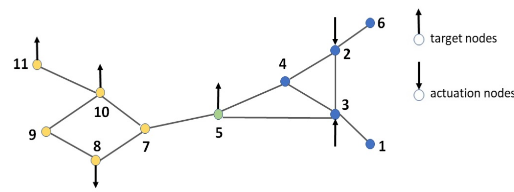

Fig. 2. Graph of the network with synchronizing dynamics from example.

Here we present a numerical example to verify the results developed in Section IV. Let us consider a network with 11 nodes whose graph is shown in Fig. 2. Here the graph is undirected and all the graph edge weights are assumed to be 1. Assume that Nodes $\{2, 3\}$ are the actuation nodes and Nodes $\{5, 8, 10, 11\}$ are the target nodes. Note that vertex 5 is a cut-vertex which separates actuation and target nodes. Therefore the regional network graph is consisted of Nodes $\{1, 2, 3, 4, 5, 6\}$ where Nodes $\{2, 3\}$ and Node 5 are actuation nodes and target node respectively. Now we seek to build controllers at Nodes $\{2, 3\}$ for which the closed-loop regional synchronization network becomes uncontrollable for the actuation at Node 5. We use our algorithm to build such controllability blocking controller. Among the eigenvalues of $\tilde{\mathbf{L}}$ we choose $\tilde{\lambda}_p = 2.7459$ in Step 1. In Step 3 we choose the right eigenvector associated with the zero eigenvalue of $\tilde{\mathbf{L}}$ to be modified. As obtained $\mathbf{V}$ is invertible, from (8) we obtain $\tilde{\mathbf{F}} = [4.081, 4.081, 4.081, 4.081, 4.081, 4.081; -4.081, -4.081, -4.081, -4.081, -4.081, -4.081]$. By checking we confirm that all the eigenvalues and right eigenvectors of $\tilde{\mathbf{L}}$ except $\tilde{\mathbf{v}}_{c_1}$ are maintained in $(\tilde{\mathbf{L}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}})$ and, the mode $(-\tilde{\lambda}_p)$ is uncontrollable at Node 5 in the regional synchronization network model as dictated by Theorem 1. We further note that $(-\tilde{\lambda}_p)$ is also an uncontrollable mode in the closed-loop base synchronization model i.e. $(-(\mathbf{L} + \mathbf{BF}))$ for the actuation at any of the nodes from $\{5, 7, 8, 9, 10, 11\}$ as stated by Theorem 2 where $\mathbf{F} = [4.081, 4.081, 4.081, 4.081, 4.081, 4.081, 0, 0, 0, 0, 0; -4.081, -4.081, -4.081, -4.081, -4.081, -4.081, 0, 0, 0, 0, 0]$. Note that the entries of $\mathbf{F}$ corresponding to the Nodes $\{7, 8, 9, 10, 11\}$ are zero. However the eigenvalues of $\mathbf{L}$ are not maintained in $(\mathbf{L} + \mathbf{BF})$. By checking we find that $(-(\mathbf{L} + \mathbf{BF}))$ is still a stable matrix; hence in this example we are not required to use time-scale separating control.

VI. Conclusions

The design of local control systems for network synchronization processes to block controllability at remote nodes has been considered. Based on a joint eigenvalue-right eigenvector assignment technique an algorithm has been developed which works under broad conditions and maintains the open-loop eigenvalues and a subset of right eigenvectors. A regional feedback solution has been obtained, by exploiting the topology of the network graph and time-separating principle. The developed algorithms can be considered as preliminary step toward practical techniques for achieving adversarial control in dynamical networks, which limits adversarial ability to direct network's dynamics. Future directions of this works include: 1) generalization of the algorithm for more relaxed conditions and for more complex network models, and 2) extension of the algorithm to allow blocking controllability of multiple critical modes subject to adversarial interests.

Acknowledgements

The authors gratefully acknowledge the support of the United States National Science Foundation under grants CNS-1545104 and CMMI-1635184.

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