Pole and Residue Estimation from Impulse Response Data: New Error Bounding Techniques

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Abstract—Estimates of nonrandom pole and residue parameters from noisy impulse-response data are characterized. Specifically, Barankin-type lower bounds (BB) on the estimation error variance of unbiased estimators are developed for single-input single-output systems with multiple but distinct real poles. Two variants of the Barankin-type bound are compared with the widely-used Cramer-Rao lower bound (CRB) in examples. The BB is found to significantly improve on the CRB when noise levels are high compared to the impulse response signal, indicating limited effectiveness of the CRB for unbiased estimators in this low signal-to-noise regime. In addition, an apparent paradox in the error bounds for low signal-to-noise ratios (SNR), which arises because the bounds are limited to unbiased estimators, is explored. This paradox suggests the use of biased estimators for pole estimation specially in low SNR settings. Two simple constructions of biased estimators are developed, which demonstrate lower mean square errors than the BB in low SNR regimes in examples.

Index Terms—Estimation, Identification, Stochastic systems, Barankin bound.

I. INTRODUCTION

The problem of estimating non-random model parameters of linear time-invariant systems (e.g., poles, residues, zeros) from noisy impulse response data [1]–[8] has been of interest to signal processing and controls engineers. Beyond building estimators, there is a considerable interest in obtaining bounds on the best achievable estimator performance, as a means to gauge estimator performance and provide confidence intervals on estimates. The Cramer-Rao bound (CRB) is popularly used as a lower bound on error variances because it is simple to compute yet tight in some circumstances; given these advantages, the CRB for linear-system parameter estimates has been extensively studied [1], [2], [3]. These bounds have recently found concrete application as metrics for estimator performance in infrastructure monitoring (e.g., oscillation monitoring in the power grid, resonance analysis in flexible structures) [9]–[11].

However, even in the limit of a long data horizon, the Cramer-Rao bounds (CRB) are not guaranteed to be tight for the problem of estimating linear-system parameters from noisy impulse response data, as the regularity conditions needed for tightness are typically not satisfied. In our preliminary studies, we explored the potential lack-of-tightness of the CRB [12], [13]. We did so by computing a tighter bound, specifically a simplified Barankin-type bound called as Hammersley-Chapman-Robbins lower bound (HCRB) [14], [15], first for a system with a single pole [12] and then for a system with multiple but distinct real poles [13] (where the test vectors’ directions were assumed to be along the standard basis vectors in the parameter space). These analyses demonstrated a gap between the CRB and HCRB, hence indicating that CRB is not tight; further, the gap was found to be large in the case of low signal-to-noise ratio (large measurement noise or small residue). The lack of tightness of the Cramer-Rao bound is concerning for infrastructure-monitoring applications, since it may lead to overly-optimistic confidence intervals.

One main contribution of this work is to pursue computation of more general Barankin-type bounds (BB) on error variances of pole and residue estimates, for multi-dimensional linear-system models. Barankin bounds [16] are appealing because they are known to provide tight bounds under high noise settings, and can be guaranteed to be tight for much relaxed regularity conditions. However, because of the complexity of the bound calculation, Barankin-type bounds for linear-system-parameter estimates have not been computed (except in simplified form in our preliminary work). Here, we pursue computation of the BB for parameter estimates of single-input single-output linear systems with multiple non-repeated real poles, when noisy impulse response data is used for estimation. The BB expressions are developed and phrased as the minimization of a cost function that has an explicit form, whereupon optimization techniques can be used to find the BB. Examples show that there is a large gap between the general BB bound and the HCRB found in our preliminary work, particularly at low signal-to-noise ratios, and hence the CRB and HCRB are both far from tight in this regime. In addition, the examples are used to explore how the Barankin-type bounds depend on the calculation variants (number of points used, choice of basis vectors).

A second main contribution of the work is to examine an apparent paradox regarding the various lower bounds obtained for pole estimates (including the CRB, HCRB, and BB). The paradox is that, for very low signal-to-noise ratios, all three bounds become significantly large, yet recognition of stability/instability of the response immediately implies that the mean squared error is bounded by 4. We argue that the apparent paradox results from the fact that the bounds are restricted to the class of unbiased estimators. We stress that no unbiased estimator exists when pole estimates are constrained to the interval [−1, 1] for which unbiased...
estimators have to guess values beyond this interval causing large estimation errors. We present two simple constructions of biased estimators of pole and demonstrate that their mean square errors (MSEs) are much smaller than the Barankin bound in low signal-to-noise ratio (SNR). These examples suggest the need for allowing bias in estimating pole for better estimation accuracy.

The paper is organized as follows. In section II, we provide a brief review on BB and related prior works. In section III, we formulate the estimation and error analysis problems. In section IV, we derive expressions for the BB. In Section V, we consider several numerical examples to understand the tightness of the bounds. In Section VI, we discuss the apparent paradox regarding the bounds at low SNR and point out the pitfalls of unbiasedness. In Section VII we present numerical examples on biased estimators of pole. Finally we draw our conclusion in Section VIII.

II. Multi-Parameter Barankin Bound: Brief Review

The research presented here is closely related to the literature on the Barankin bound (BB). The BB is appealing because it can provide tight lower bounds on the error of unbiased estimators of non-random parameters, under quite general conditions. Nevertheless, the bound has found relatively limited application, due to the difficulty of calculating the bound either analytically or numerically. Some simplified forms of Barankin-type bounds have been applied recently in: 1) estimation of sparse non-random vectors in the presence of noise [17], [18], and 2) estimation of multiple change points in time series [19]. Barankin-type bounds are also being used in threshold prediction in direction-of-arrival (DOA) estimation and source localization [20], [21]. In our previous work, a simpler form of Barankin-type bound – the Hammersley-Chapman-Robbins lower bound (HCRB) with test vectors’ direction restricted along the standard basis vectors, was considered for pole and residue estimation from noisy impulse response data [12], [13]. This study extends the analysis to a more general class of Barankin bounds, where test vectors can be selected at will in the parameter space, and the number of test points can also be chosen.

The Chapman-Robbins version of the multi-parameter Barankin lower bound on the error covariance for a non-random parameter vector’s estimate is briefly reviewed here, based on the development in [22], [23]. Formally, consider an unbiased estimator $\hat{\theta}(y)$ for a nonrandom parameter vector $\theta = [\theta_1, \theta_2, \cdots, \theta_m]^T \in \mathbb{R}^k$, which uses a set of observations $y = [y_1, y_2, \cdots, y_n]$. The observations are modeled as random variables generated according to the joint probability density function $f(y_1 = y_1, y_2 = y_2, \cdots, y_n = y_n; \theta)$ or succinctly $f(y; \theta)$. The Barankin bound $BB_\theta$ provides a lower bound on the covariance matrix $\text{COV}\{T(y)\} = E\{(T(y) - \theta)(T(y) - \theta)^T\}$, in the sense that $BB_\theta - \text{COV}\{T(y)\}$ is negative semidefinite. The $m$-point Barankin bound is given by:

$$BB_\theta = \sup_{\phi} \left( \phi J^\dagger \phi^T \right)$$

for $1 \leq i \leq m$ and $1 \leq j \leq m$, the $(i,j)$ entry of $J$ is:

$$J_{ij} = E\left[ \frac{f_{\theta_i} f_{\theta_j}}{f_{\theta_i}} \right] - 1$$

Here, $\phi = [(\theta_1 - \theta), (\theta_2 - \theta), \cdots, (\theta_m - \theta)]$ where $\theta_1, \theta_2, \cdots, \theta_m$ are the $m$ number of different test points (or equivalently, we say $(\theta_1 - \theta), (\theta_2 - \theta), \cdots, (\theta_m - \theta)$ are $m$ number of different test vectors) in the parameter space. Note that, in (1), $\{\cdot\}^T$ denotes the Moore-Penrose pseudo inverse, and the supremum is taken over all possible test points/vectors. Also, $(\phi J^\dagger \phi^T)$ is supremized in the sense that any particular scalar quadratic form $z^T(\phi J^\dagger \phi^T)z$ (chosen as desired based on the statistic that is of interest) is supremized. We note that, in our previous work, we considered a further restricted form of the presented bound wherein the test vectors’ directions were constrained to be along the standard basis vectors. Furthermore, the number of test points $m$ considered in the analysis may either be fixed to equal the number of estimated parameters, or more generally may be chosen at will. These variants on the bounds have been identified in the literature interchangeably as Barankin Bounds (BB), Chapman-Robbins type Barakin Bounds, or Hammersley-Chapman-Robbins bounds; for simplicity, we refer to the bounds described above from here on as the Barankin bound.

The Barankin bound does not require any of the regularity assumptions of the CRB for tightness. Furthermore, the higher point $(m \geq k)$ Barakin bound is at least as tight as the CRB, with the two bounds coinciding when $m = k$ and the supremum is achieved at $m$ mutually independent test vectors in unconstrained parameter space having infinitesimally small magnitudes [14], [15].

III. Problem Formulation

Here, we study parameter estimation for a discrete-time system having distinct real poles. Specifically, a system with the transfer function $H(z) = \sum_{l=1}^{r} \frac{A_l}{1 - a_l z^{-1}}$ is considered, where the nonrandom parameters $A_l$ and $a_l$ are the real poles and residues of the system, respectively. Noisy measurements are made of the system’s response at the times $k = 0, \ldots, n$, upon impulsive stimulation at time $k = 0$. The measured impulse response can be written as:

$$y(k) = x(k) + w(k) = \sum_{l=1}^{r} A_l a_l^k + w(k)$$

for $k = 0, \ldots, n$ where $w(k)$ is assumed to be a zero-mean Gaussian white noise with variance $\sigma^2$. We note that $x(k) = \sum_{l=1}^{r} A_l a_l^k$ is the true impulse response of the system.

Our aim here is, firstly, to derive the Barankin bound on the estimation error covariance matrix for the nonrandom parameter vector $\theta = [a^T A^T]^T = [a_1 a_2 \cdots a_r A_1 A_2 \cdots A_r]^T$. Secondly, we perform
numerical computations of the bound for a simple system, to gain insight on the performance of unbiased estimators, and to compare the Barankin bound with simpler bounds such as the Cramer-Rao bound. Finally, we make comments on the large values of the bounds on pole estimates and point out possible remedies from that by suggesting biased estimators through numerical examples.

IV. ANALYSIS OF THE BARANKIN BOUNDS

In this section we derive the expression for the BB, as introduced in Section II, for the posed estimation problem. We also discuss possible simplifications of the bounds, and consider the computational complexity for different variants of the calculation (e.g., number and choice of test points). We first begin by presenting the main result in the following theorem, which gives the expression for BB for the joint estimation of poles and residues from impulse response data:

**Theorem 1** The m-point Barankin bound for the pole and residue estimation problem is:

$$BB = \sup \left( \phi J^T \phi^T \right)$$

where for $1 \leq i \leq m$ and $1 \leq j \leq m$, the $(i,j)$ entry of $J$ is given by:

$$J_{ij} = \exp \left[ \frac{1}{\sigma^2} \sum_{l=1}^{r} \sum_{p=1}^{r} \left( \frac{A_{li}A_{pj}}{a_{ij}} (a_{ij}p_{ji})^{(n+1)} - 1 \right) - \frac{1}{\sigma^2} \sum_{l=1}^{r} \sum_{p=1}^{r} \left( \frac{A_{li}a_{ij} + A_{li}a_{pj}}{a_{ij}} - 1 \right) \right] - 1$$

Here, any scalar quadratic form $z^T(\phi J^T \phi^T)z$ may be supremized to obtain a bound. Note, $A_{ij}$ and $a_{ij}$ denote the value taken by $i$-th test point for $A_{ij}$ and $a_{ij}$ respectively.

**Proof:**

Since the noise is zero mean, white, and Gaussian with variance $\sigma^2$, the joint probability density function (pdf) for $y(k)$, $k = 0, 1, 2, \cdots, n$, can be written as:

$$f(y; \theta) = \frac{1}{(2\pi\sigma^2)^{n+1}} \exp \left( -\frac{1}{2\sigma^2} \sum_{k=0}^{n} [y(k) - \sum_{i=1}^{r} A_{ik}]^2 \right)$$

Substitution of the joint pdf into the definition of $J_{ij}$ yields:

$$J_{ij} = \{E_{\phi}\{\exp \left( \sum_{k=0}^{n} C_k y(k) + i_j D_k \right) \} \} - 1$$

where $C_k = \frac{1}{\sigma^2} \sum_{i=1}^{r} \sum_{p=1}^{r} \left( A_{li}a_{pj} + A_{li}a_{pj} - 2A_{li}a_{pj} \right)$ and $D_k = \frac{1}{\sigma^2} \sum_{i=1}^{r} \sum_{p=1}^{r} 2(A_{li}a_{pj}) + A_{li}a_{pj} - (A_{li}a_{pj})(A_{pi}a_{pj}) - (A_{li}a_{pj})(A_{pi}a_{pj}) .

Since each observation is Gaussian, the exponent $u_{ij} = \sum_{k=0}^{n} (C_k y(k) + i_j D_k)$ in the expression for $J_{ij}$ is also Gaussian. The mean of $u_{ij}$, denoted as $\mu_{u_{ij}}$, can be evaluated as $\mu_{u_{ij}} = \sum_{k=0}^{n} (C_k y(k) + i_j D_k)$ similarily, the variance of $u_{ij}$, denoted as $\sigma^2_{u_{ij}}$, can be evaluated as $\sigma^2_{u_{ij}} = \sum_{k=0}^{n} C_k^2 \sigma^2$. Thus, using the moment generating function for the Gaussian distribution, we can simplify $E_{\phi}\{\exp (u_{ij}) \}$ as $E_{\phi}\{\exp (u_{ij}) \} = \exp (\mu_{u_{ij}} + \frac{\sigma^2_{u_{ij}}}{2}) = \exp \left[ \frac{1}{\sigma^2} \sum_{k=0}^{n} \sum_{i=1}^{r} \sum_{p=1}^{r} \left( \frac{A_{li}a_{pj}}{a_{ij}} + A_{li}a_{pj} - (A_{li}a_{pj})(A_{pi}a_{pj}) \right) \right]$. Substituting this expression into the expression for $J_{ij}$ and using formula for the summation of geometric series, we recover the expression for $J_{ij}$ in the theorem statement.

Note that BB expressions when only poles are estimated (residues are known) and when only residues are estimated (poles are known) also readily follow from Theorem 1. We call these estimation problems as pole-only estimation and residue-only estimation. For the pole-only estimation problem, all pole values may vary among the test points ($A_{li} = A_l$ for $1 \leq i \leq m$). Similarly, for the residue-only estimation problem, only residue values may vary among test points ($a_{ii} = a_i$ for $1 \leq i \leq m$) in the theorem statement. In the residue-only estimation case, the CRB and m-point BB are identical when $m \geq r$. Further, the bounds are tight in the sense that the maximum-likelihood estimator of the residues achieves the bound. Verifying this simply requires noting that the measurements $y(k)$ are a linear function of the parameters to be estimated, subject to additive Gaussian noise with fixed variance. Drawing on standard results for the CRB in the linear Gaussian case, it follows that the CRB is achieved by the maximum likelihood estimate. Since the m-point BB is guaranteed to be tighter than the CRB when $m \geq r$, it follows also that the two bounds are identical. For the sake of completeness, we present the result for residue-only estimation below as Lemma 1. However for pole estimates we notice a very significant gap between BB and CRB in the presence of noise. We discuss more on the error variance on pole estimates in the context of examples in Section V.

**Lemma 1** For the residue-only estimation problem, the m-point BB is identical to CRB for any $m \geq r$, where $r$ is the number of residue parameters to be estimated.

The expressions for the BB in Theorem 1 are useful as a starting point for numerical computation. Specifically, these expressions give an explicit closed-form expression of the Barankin information matrices, and hence the BB can be computed by optimizing a quadratic form of the inverse of a matrix which has closed-form expressions for the entries. We note that many simplification can be made to the Theorem 1 to facilitate the optimization and thus computation of the lower bound. In particular, the m-point Barankin bound is the supremum of any lower bounds obtained using any $m$ test points/vectors in the parameter space. Simper but less tight lower bound can be thus obtained by limiting the number of test points and the choice of test points/vectors. The higher the number of test points, the tighter the BB.


becomes. Due to computational complexity issues, however, it is often useful in practice to keep the number of test points smaller. As an example, the case where the number of test points is \( m = 2r \) and the test vectors are parallel to the standard basis vectors in \( \mathbb{R}^{2r} \) was considered in our previous work [12], [13], where it was referred to as a Hammersley-Chapman-Robbins bound (HCRB) which we call now restricted Barankin bound. This bound was found to be tighter than CRB (particularly at low SNR), but may not be as tight as the true Barankin bound with more points and flexibility in the test vectors. The simpler/restricted bound, however, is appealing computationally as it greatly simplifies the optimization.

V. NUMERICAL EXAMPLES ON BOUND COMPUTATION

In this section, numerical computations of the BB and CRB are undertaken in examples, to gain further insight into the gaps between the bounds. Due to the complexity of the BB calculation, we restrict ourselves to the single-pole case (\( r = 1 \)). First we present simulation results for pole-only estimation, and then we present further results on joint-estimation of both pole and residue.

As a first experiment, we fix the pole \( a_1 \) and residue \( A_1 \) at 0.8 and 0.9, and vary the noise variance. A long time horizon \( (n = 10000) \) is assumed. In Figure 1, we plot the 2-point BB, the 1-point BB (which is also identical to the single parameter HCRB [12], [14]), and the CRB as a function of the signal-to-noise ratio (where we calculate the SNR as \( \text{SNR} = \frac{1}{2\pi} \sum_{k=0}^{n} x(k)^2 \) [1]. For high noise levels (SNR < 0 dB), there is a large gap between 2-point BB and the other bounds, indicating that HCRB and CRB are not tight bounds for pole estimates. As we know that increasing the number of points tighten the Barankin bound, therefore one may expect to obtain larger BB by using more points at the expense of additional computational effort. However, for this example it is noticed that all the \( m \)-point BB where \( m \geq 2 \) are almost same which suggests that increasing the number of points in BB does not necessarily mean significant improvement of the bounds and thus optimal number of points should be used to reduce the computational burden.

Next, we determine lower bounds on the error variance for joint estimation of the pole and residue as a function of the signal-to-noise ratio (for the same pole and residue values as in the first experiment). In Figure 2, the 2-point BB, restricted 2-point BB where the test vectors’ direction are constrained along the standard basis vectors in \( \mathbb{R}^2 \) (mentioned as HCRB in [12], [13]) and CRB are shown as a function of the SNR. We notice that for the residue estimates, the 2-point BB and the restricted 2-point BB [13] are almost the same, and the gap between BB and CRB is also small, for any SNR. This suggests that the CRB may provide good characterizations of the lower bound on error variance of residue estimates in joint pole-residue estimation problem. Meanwhile, the gap between the 2-point BB and the other bounds (both the restricted 2-point BB [12], [13] and the CRB) are extremely large for pole estimates, at high noise levels (small SNR). This indicates that the Barankin bound is highly sensitive to the direction of the test vectors; and CRB and restricted 2-point BB do not yield accurate lower bounds on error variance of pole estimates. Comparing the gap between BB and CRB in Figure 1 and Figure 2, we notice that as the dimension of parameter space increases, the gaps between the bounds become larger. Hence, for multiple-pole systems, we conjecture that the gap will be even more significant, and use of CRB becomes a greater concern.

Finally, we study the dependence of the bounds on the location of the pole, for the joint pole-residue estimation example. To do so, we fix the noise variance and residue \((\frac{A_1}{a_1} = 2)\), and vary the pole \( a_1 \) from 0.1 to 0.9 (see Figure 3). The simulations show that the gaps between the 2-point BB and the other bounds are comparatively larger for fast modes (which have small signal content), which indicates that fast modes (i.e. poles having small values) are difficult to estimate, and further that the simpler bounds are not tight. A similar gap among the bounds is observed when the difference between two poles becomes small, in a multi-pole system.

VI. AN APPARENT PARADOX: PITFALL OF USING AN UNBIASED ESTIMATOR

One interesting observation from the numerical examples given in Section V is that all of the lower bounds, including the 2-point BB, the restricted BB (mentioned as HCRB in [12], [13]), and the CRB become unboundedly large at very low SNR, even in the limit of a long data horizon. This is disconcerting, as the system displays an obvious dichotomy between a stable and an unstable response. Thus, if the pole to be estimated is stable, an estimator can trivially be developed whose mean square error is bounded by 4; meanwhile, if the pole is unstable, then a very low estimation error should be achieved, as the exponential growth rate in the response allows precise estimation. Given this, it seems that the mean square error should be bounded by 4 at all
Joint Estimation: Lower Bounds on Error Variance of Pole

Fig. 2. Lower bounds on error variance for joint-estimation problem as a function of SNR with $a_1 = 0.8$ and $A_1 = 0.9$.

Joint Estimation: Lower Bounds on Error Variance of Residue

Fig. 3. Lower bounds on error variance for the pole-estimation problem as a function of pole location, with $A_1 = 0.6$ and $\sigma = 0.3$.

parameter values, and the arbitrarily-large bound appears to be paradoxical.

The apparent paradox results because the bounds considered in this work so far, including the Barankin-type bounds and the Cramer-Rao bound, are restricted to the class of unbiased estimators. When unbiased estimators are sought and responses for nearby parameter values are hard to distinguish (i.e. the derivative of the PDF with respect to the parameter is small), then the estimator error necessarily becomes large. This is because maintaining unbiasedness for this range of hard-to-distinguish parameter values requires that the estimator egregiously over-guesses or under-guesses the parameter values. When in addition the parameters are restricted or can be constrained to an interval (as is the case in the pole-estimation problem because of the distinction between stable and unstable responses), an apparently paradoxical situation results where a limited-error (but biased) estimate can readily be found.

This notion can be formalized using results which demonstrate that unbiased estimators cannot be found for many estimation problems, such that interval constraints on the estimated parameters are respected (see [25]). As the Gaussian distribution described by (4) is absolutely continuous with respect to the poles, Theorem 1 of [25] applies, and we can conclude that no unbiased estimator of the pole exists such that the estimates are restricted to $[-1, 1]$ for all parameter values. Similar conclusion can be made applying Theorem 2 of [25] when pole estimates is restricted to open interval (i.e. $(-1, 1)$). Therefore any unbiased estimator will necessarily guess unstable pole values to maintain unbiasedness, which results in large estimation error at low SNR. Due to importance of this observation, we mention it as a lemma below which follows from the results of [25].

**Lemma 2** For the pole and residue estimation problem or pole-only estimation problem, no unbiased estimator can be found such that the pole estimates are constrained to the interval $[-1, 1]$.

VII. Numerical Examples on Biased Estimators for Pole Estimation

The discussion in Section VI motivates us to study biased estimators for poles, as unbiased estimators have unwarrantedly-large estimation error when SNR becomes low. In this section, we provide two simple constructions of biased estimators for the pole-only estimation problem, in the context of numerical example: 1) a constrained estimator and 2) a biased estimator whose bias is linear with respect to the pole. Our linearly biased approach instantiates the theoretical developments of biased estimators given in [26], [27] for the pole estimation problem. [27] discusses on linearly biased estimators and suggests methods for finding suitable linear bias vector specifically for estimation problem where CRB is quadratic with respect to parameter vector to be estimated via convex optimization technique (However in our problem CRB is not quadratic with respect to parameter).
In the numerical examples, we consider a single pole system \((r = 1)\) for simplicity; however the approaches apply for multi-pole systems. For this simple system we demonstrate that the constructed biased estimators can achieve much lower mean square error (MSE) than the Barankin bound of unbiased estimator.

![Bias of the Estimator](image)

**Fig. 4.** Bias of the constrained estimator \(\hat{a}_{cb}\) as a function of pole with \(A = 0.9\) for different noise variances.

![Mean Square Error (MSE) of MLE and Biased Estimator (BE)](image)

**Fig. 5.** Mean square errors of MLE and the constrained estimator \(\hat{a}_{cb}\) as a function of SNR with \(A = 0.9\), \(\sigma = 0.8\) and noise is varied. (Recall that MLE achieves CRB in this case and, MSE and error variance are identical for unbiased estimator. Thus MSE of MLE also represents the lower bounds on corresponding MSE). Here \(A \geq 0\) and noise is varied. (Recall that MLE achieves CRB in this example). From Figure 5, it is evident that at low SNR the gaps between the MSE of the MLE \(\hat{a}_{MLE}\) and the MSE of \(\hat{a}_{cb}\) are significant. We also see that the mean square error achieved by the constrained estimator is limited to 4 even for small SNR, which matches with intuition for the pole estimation problem. These comparisons confirm the importance of allowing bias for the estimation of pole in low SNR setting. This constrained approach can be used for any number of observations \((n > 1)\), any number of poles \((r > 1)\) and for any unbiased estimators (not necessarily MLE); however exact computation of MSE of the constrained estimator may not be always possible though boundedness of MSE is guaranteed.

Second, we consider a linearly biased estimator of pole--i.e., an estimator whose bias is linear with pole. Therefore we now consider a biased estimator whose bias is given by \(ma\) where \(m\) is a constant. The usefulness of using linear bias is that the biased estimator is readily obtained as: \(\hat{a}_{lb} = (1 + m)\hat{a}_u\) where \(\hat{a}_u\) is an unbiased estimator. Also note that in Figure 4 the bias of the estimator \(\hat{a}_{cb}\) becomes almost linear with respect to the pole at small SNR, suggesting that a linearly biased estimator may be effective yet computationally friendly. The MSE of \(\hat{a}_{lb}\) is given by: \(MSE\{\hat{a}_{lb}\} = (1 + m)^2 MSE\{\hat{a}_u\} + m^2a^2\). We can utilize prior knowledge on the system’s stability to solve a minimax optimization problem to obtain an optimum value for \(m\). With the goal of reducing MSE over all allowable values of pole, we select an optimum value for \(m\) as \(m^* = \arg \min_{m} \max_{a \in L} (MSE\{\hat{a}_{lb}\} - MSE\{\hat{a}_u\})\) where \(L\) is the allowable set of values of \(a\).

We plot the MSE of linearly biased estimator \(\hat{a}_{lb}\) in Figure 6 where we assume that the unbiased estimator \(\hat{a}_u\) achieves the Barankin bound (Thus the MSEs shown in Figure 6 represent the lower bounds on corresponding MSE). Here we use \(A = 0.9\), \(n = 10000\) and assume that pole is restricted to \([-0.95, 0.95]\) for the minimax optimization. Note that the Barankin bound shown in the figure is actually the 2-point Barankin bound, as we know it is tight from Section V. The figure shows that the MSE of the linearly

\[
\hat{a}_{cb} = \begin{cases} 
\frac{y(1)}{A} & \text{if } -1 < \frac{y(1)}{A} < 1, \\
1 & \text{if } \frac{y(1)}{A} \geq 1, \\
-1 & \text{if } \frac{y(1)}{A} \leq -1.
\end{cases}
\]
biased estimator is almost the same as the BB at high SNR ($\sigma = 0.7$). However, at low SNR ($\sigma = 2$), the linearly biased estimator can achieve a MSE that is much lower than the BB, demonstrating the usefulness of the linearly biased estimator. However the reduction in the lower bound using the linearly biased estimator depends on the MSE/error variance of the chosen unbiased estimator, and the allowable values of pole in the minimax optimization. We leave it to future work to examine these dependencies in detail.

and hence unbiased estimation is ineffective. This suggests the need for biased estimators. Two simple constructions of biased estimators are provided, and are shown to achieve smaller MSEs with respect to the BB at low SNRs for a one-pole system. This suggests that unbiased estimators should be used with caution for the pole-estimation problem, and attention should given to the development of biased estimators and associated performance bounds.

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