Homework 1 (Cpt S 223)

Due Date: September 7, 2012

Total points: 43

In this homework, we will review problems from topics in math including proving techniques, solutions to recurrences and other mathematical forms, and recursion.

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1. (4 points)

   Write a simple recursive function to calculate (and return) the height of a binary tree[1] T.

   The *height* of a tree T is defined as the number of levels below the root. In other words, it is equal to the length of the longest path from the root (i.e., number of edges along the path from the root to the deepest leaf). Note that the term "nodes" is used to include both internal nodes and leaf nodes. You can assume the following tree node structure:

   class Node {

   *Node *left;* *// points to the left subtree*

   *Node *right;* *// points to the right subtree*

   }

   Your answer can be in C++ syntax *or* in the form of a generic pseudocode (preferred).

2. (5 points)

   Evaluate the series: $\Sigma_{i=0}^{\infty} \frac{2\times i}{4^i} = ?$

   Show all the steps while deriving the answer.

   Hint: A technique for a similar problem is described in section 1.2.3 (page 4, Weiss).

3. (5 points)

   Solve the following mathematical recurrence:

$$\mathcal{T}(n) = \begin{cases} 1 & , n = 1 \\ 2\,\mathcal{T}(\frac{n}{2}) + n & , n > 1 \text{ and n is a power of 2} \end{cases} \quad (1)$$

   (Note: Your answer should provide a closed form for $\mathcal{T}(n)$.)

[1] Note that nothing has been specified about this being a *complete* binary tree. So it is *not* mandatory that every internal node must have two children or that its a balanced tree.

4. (5 points)

   A *complete k-ary tree of depth D* is one in which: i) there are exactly k children for every internal node; ii) there are D levels below the root; and iii) each level is completely saturated. Note for example that if k=2 then it is a complete binary tree, $k = 3$ means a complete ternary tree and so on.

   For a given value of k, prove that the number of the nodes in a complete k-ary tree of level D is: $\frac{k^{D+1}-1}{k-1}$.
   You can either use induction (on D) or provide a proof by construction — i.e., directly deriving the expression, whichever suits you.

5. (6 points)

   a) Using the proof by contradiction technique, prove the following lemma:

   Lemma 1: If n^2 is odd then n is odd.

   b) Using the above result, show the following corollary:

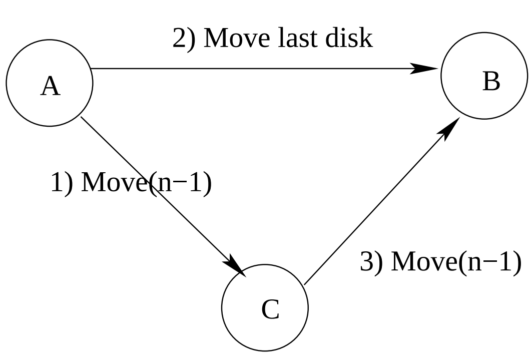
   Corollary 1: If n^2 is odd then n^3 is odd.

6. (8 points)

   Rewrite the pseudocode presented in class (slide #37) for the Tower of Hanoi problem *without* tail recursion.

7. (10 points)

   We discussed an algorithm to solve the Tower of Hanoi problem in class. The approach we discussed for optimally moving n disks from peg A to peg B using peg C can be illustrated as shown below:



* Steps are numbered from 1 to 3.

Therefore, number of moves in Move(n) is:

2 * Move(n−1) + 1

Now consider the following variant of the Tower of Hanoi problem. To the original problem, let us add a new condition: Each disk move now must be performed only in the *clockwise* direction — ie., $A \rightarrow B$, $B \rightarrow C$, or $C \rightarrow A$. So, for example, if I have to move a disk from A to C then that will involve a minimum of two moves (one from A to B and then from B to C); but from C to A needs only one move. All the old conditions from the original version of the problem still apply.

Under this new problem formulation, let $Q(n)$ denote the minimum number of moves required to transfer n disks from $A \rightarrow B$ using C. Also, let $R(n)$ denote the minimum number of moves to transfer n disks from $B \rightarrow A$ clockwise using C.

Prove that:

$$\mathcal{Q}(n) = \begin{cases} 0 & , n = 0 \\ 2R(n-1) + 1 & , n > 0 \end{cases} \tag{2}$$

$$\mathcal{R}(n) = \begin{cases} 0 & , n = 0 \\ Q(n) + Q(n-1) + 1 & , n > 0 \end{cases} \tag{3}$$

Note: to provide the proof you need to come up with an algorithm to transfer n disks from say A to B using C, but under the new clockwise condition. Provide your answer in the form of an illustration similar to the figure shown above for the original version of the problem.