Undecidability

Reading: Chapter 8 & 9

Decidability vs. Undecidability

- There are two types of TMs (based on halting):

  (*Recursive*)

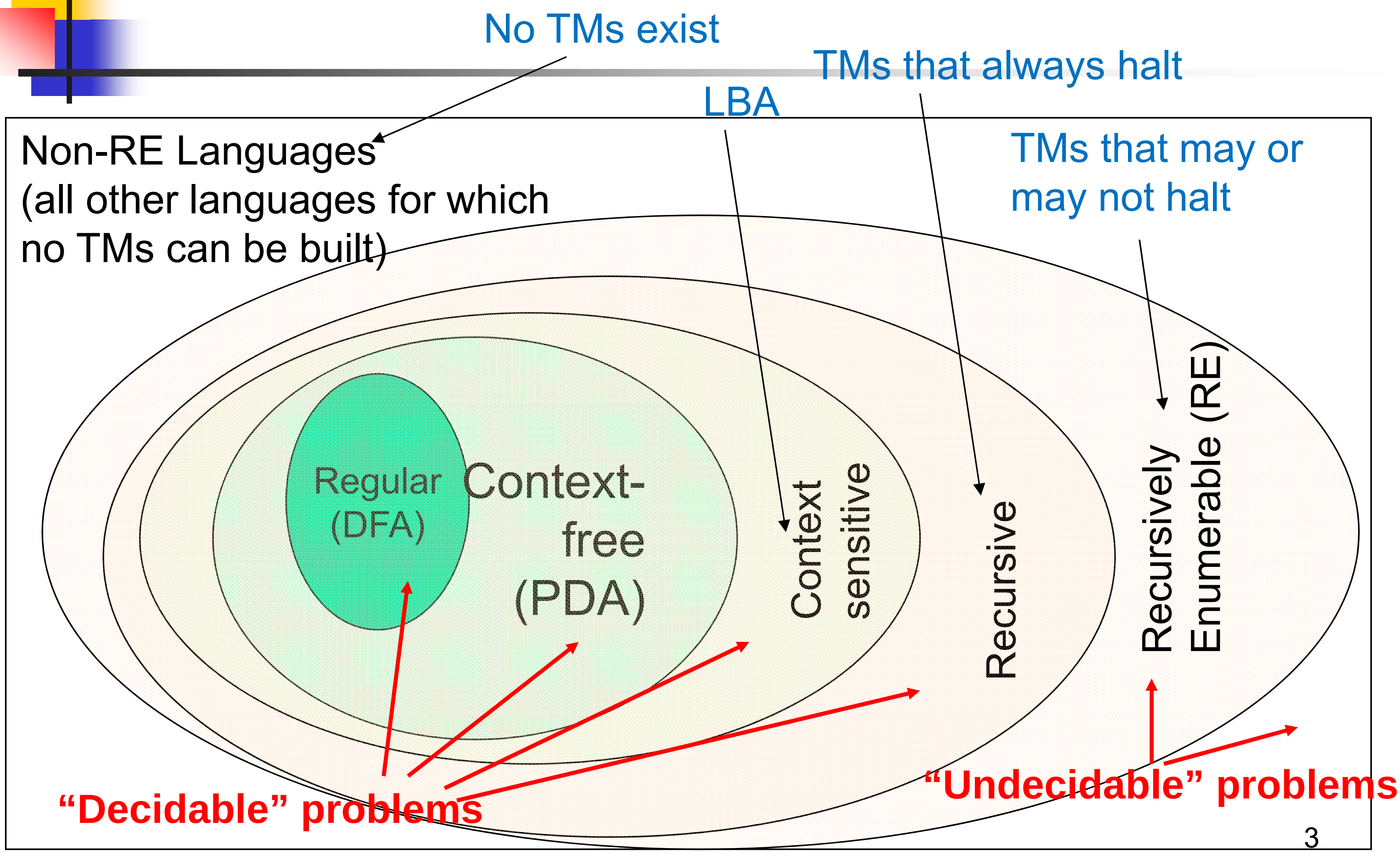
  TMs that *always* halt, no matter accepting or non-accepting ≡ **DECIDABLE** PROBLEMS

  (*Recursively enumerable*)

  TMs that *are guaranteed to halt* only on acceptance. If non-accepting, it may or may not halt (i.e., could loop forever).

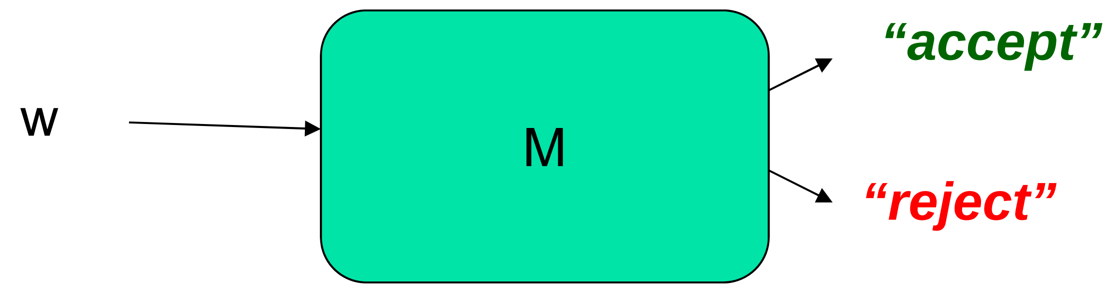
- **Undecidability:**
  - Undecidable problems are those that are not recursive

Recursive, RE, Undecidable languages



Recursive Languages & Recursively Enumerable (RE) languages

- Any TM for a <u>Recursive</u> language is going to look like this:

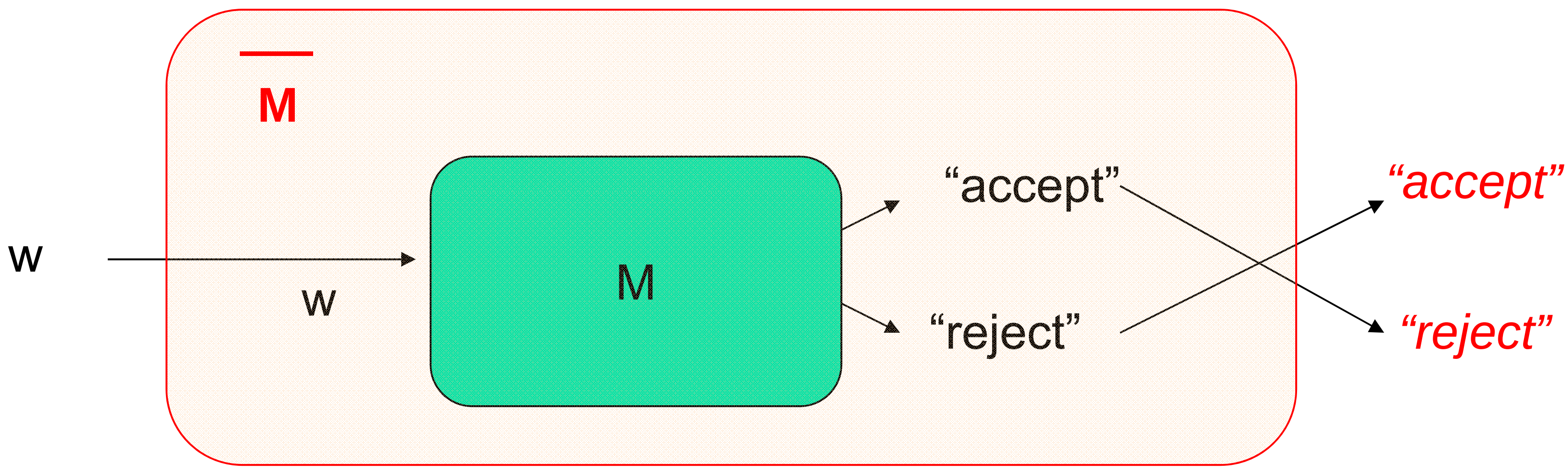


- Any TM for a <u>Recursively Enumerable</u> (RE) language is going to look like this:

Closure Properties of:
- the Recursive language class, and
- the Recursively Enumerable language class

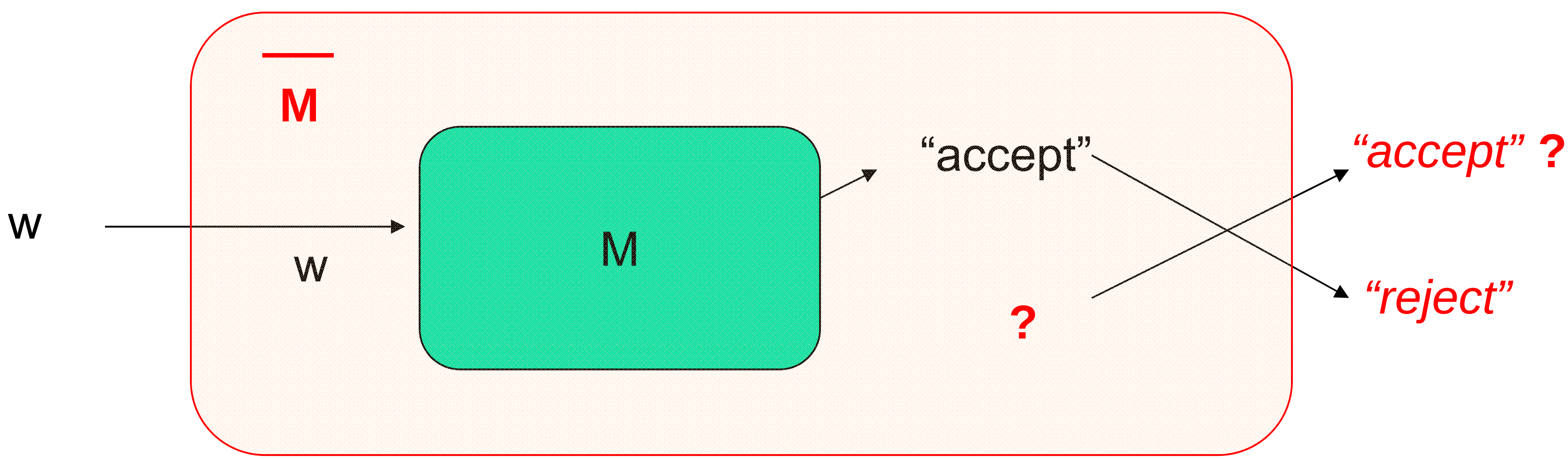
Recursive Languages are closed under complementation

- If L is Recursive, $\overline{L}$ is also Recursive



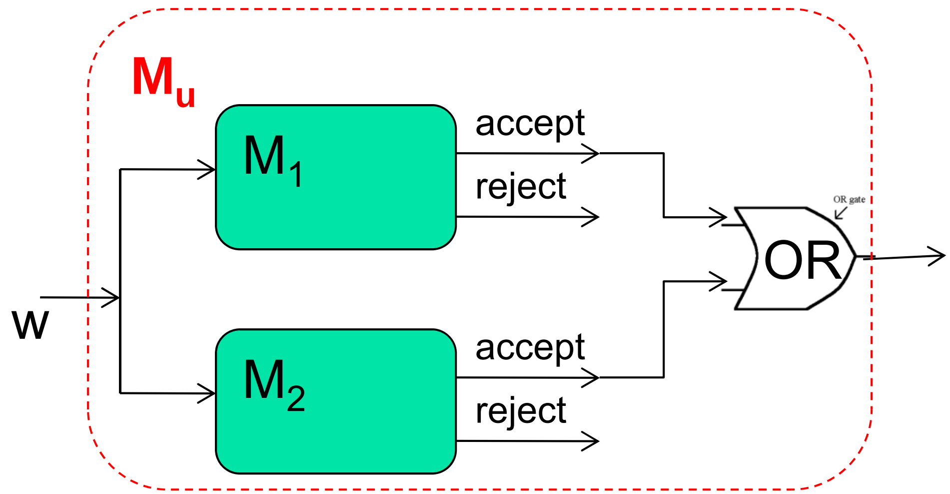
Are Recursively Enumerable Languages closed under complementation? (NO)

- If L is RE, $\overline{L}$ need not be RE



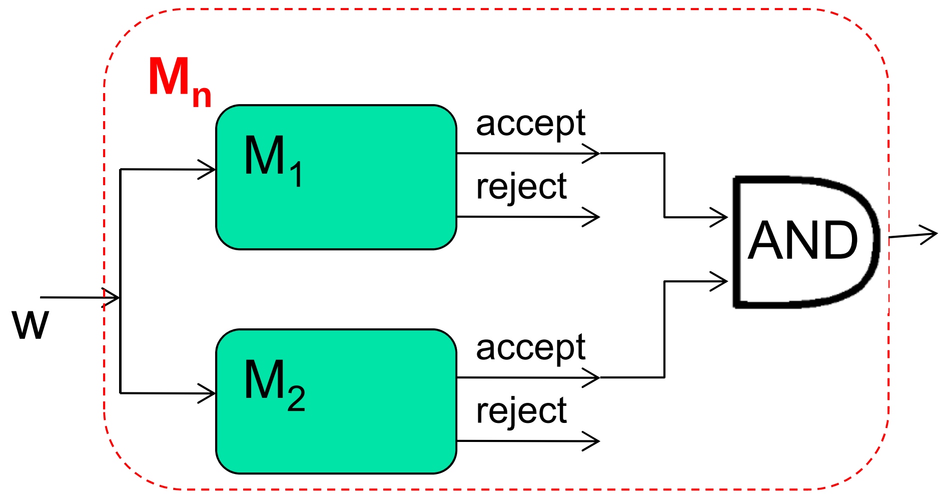
Recursive Langs are closed under Union

- Let M_u = TM for $L_1 \cup L_2$
- M_u construction:
  1. Make 2-tapes and copy input w on both tapes
  2. Simulate M_1 on tape 1
  3. Simulate M_2 on tape 2
  4. If either M_1 or M_2 accepts, then M_u accepts
  5. Otherwise, M_u rejects.



Recursive Langs are closed under Intersection

- Let M_n = TM for $L_1 \cap L_2$
- M_n construction:
  1. Make 2-tapes and copy input w on both tapes
  2. Simulate M_1 on tape 1
  3. Simulate M_2 on tape 2
  4. If M_1 AND M_2 accepts, then M_n accepts
  5. Otherwise, M_n rejects.



Other Closure Property Results

- Recursive languages are also closed under:
  - Concatenation
  - Kleene closure (star operator)
  - Homomorphism, and inverse homomorphism
- RE languages are closed under:
  - Union, intersection, concatenation, Kleene closure

- RE languages are *not* closed under:
  - complementation

“Languages” vs. “Problems”

A “language” is a set of strings

Any “problem” can be expressed as a set of all strings that are of the form:

- “<input, output>”

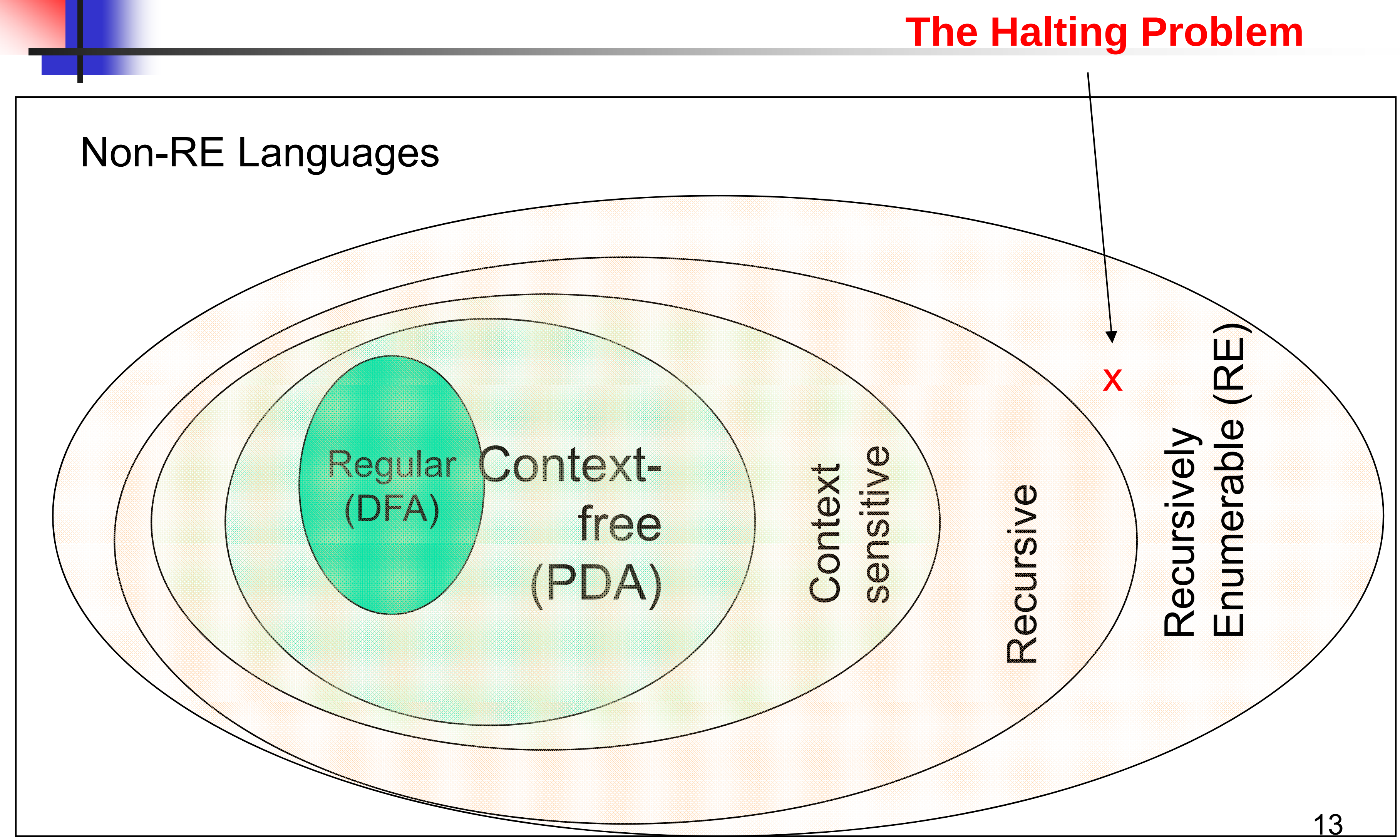
e.g., Problem (a+b) ≡ Language of strings of the form { “a#b, a+b” }

==> Every problem also corresponds to a language!!

Think of the language for a “problem” == a *verifier* for the problem

The Halting Problem

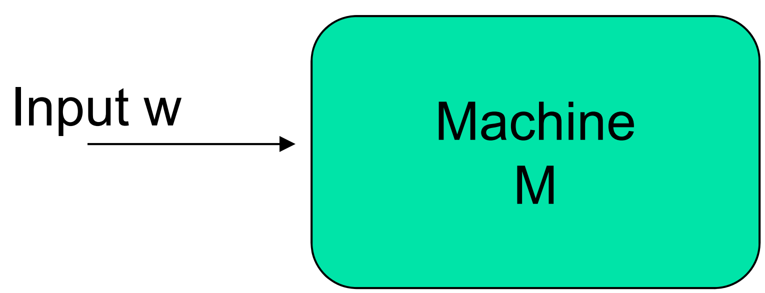
An example of a <u>recursive enumerable</u> problem that is also <u>undecidable</u>


The Halting Problem
Non-RE Languages
Regular
(DFA)
Context-
free
(PDA)
Context
sensitive
Recursive
Recursively
Enumerable (RE)
x

What is the Halting Problem?

<u>Definition of the “halting problem”:</u>

- *Does a givenTuring Machine M halt on a given input w?*



A Turing Machine simulator

The Universal Turing Machine

- <u>Given:</u> TM **M** & its input **w**
- <u>Aim:</u> Build another TM called “H”, that will output:
  - “*accept*” if M accepts w, and
  - “reject” otherwise

- An algorithm for H:
  - Simulate M on w

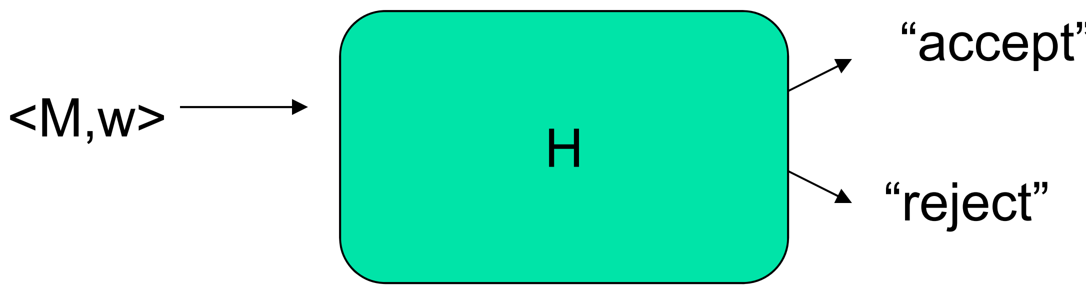
Implies: H is in RE

  - $$H(<M,w>) = \begin{cases} \textit{accept}, & \text{if } M \text{ accepts w} \\ \textit{reject}, & \text{if M does does not accept w} \end{cases}$$

<u>Question:</u> If M does *not* halt on w, what will happen to H?

A Claim

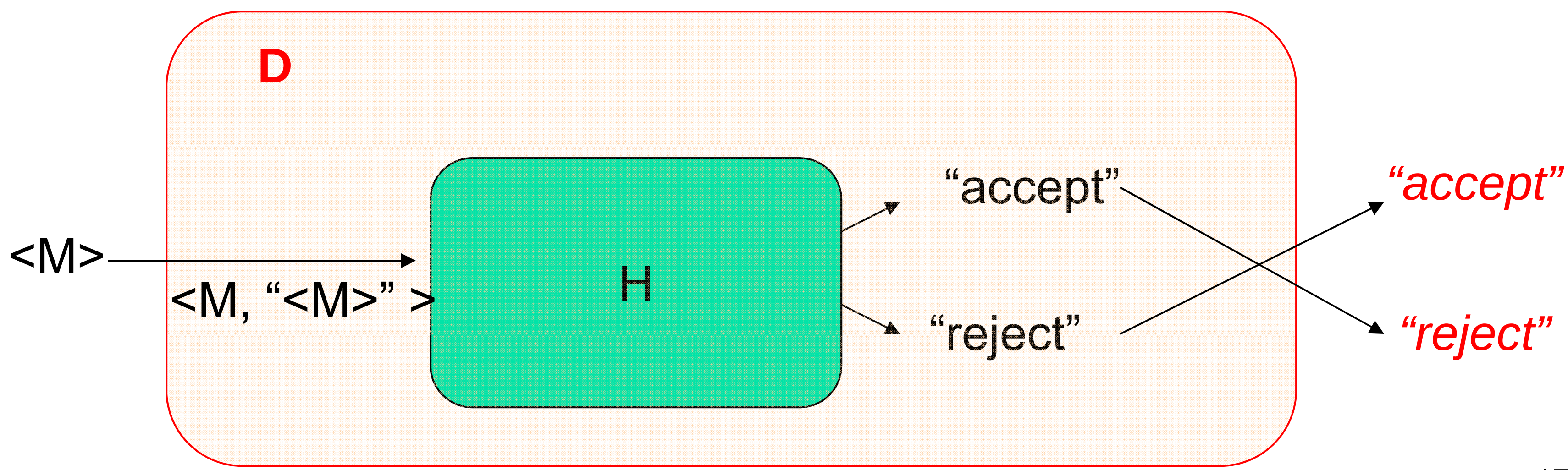
- <u>Claim:</u> No H that is always guaranteed to halt, can exist!
- <u>Proof:</u> (Alan Turing, 1936)
  - By contradiction, let us assume H exists



Therefore, if H exists ➔ D also should exist.
But can such a D exist? (if not, then H also cannot exist)

HP Proof (step 1)

- Let us construct a new TM **D** using H as a subroutine:
  - On input <M>:
    1. Run H on input <M, <M> >;   //(i.e., run M on M itself)
    2. Output the *opposite* of what H outputs;



HP Proof (step 2)

- The notion of inputing “<M>” to M itself
  - A program can be input to itself (e.g., a compiler is a program that takes any program as input)

$$D\,(\langle M \rangle) = \begin{cases} \textit{accept}, & \text{if M does } \textit{not} \text{ accept } \langle M \rangle \\ \textit{reject}, & \text{if M accepts } \langle M \rangle \end{cases}$$

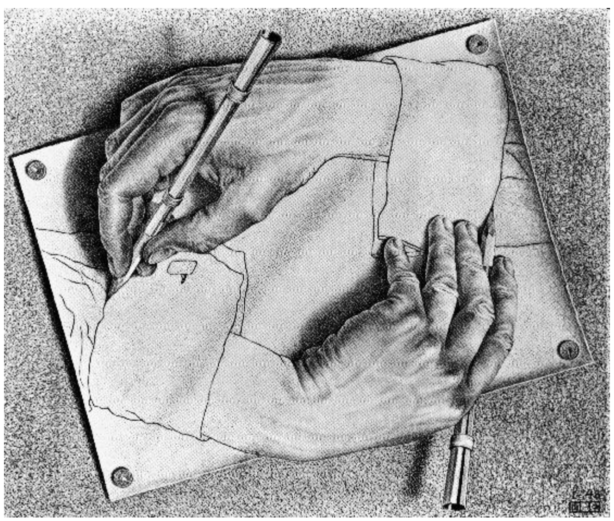
Now, what happens if D is input to itself?

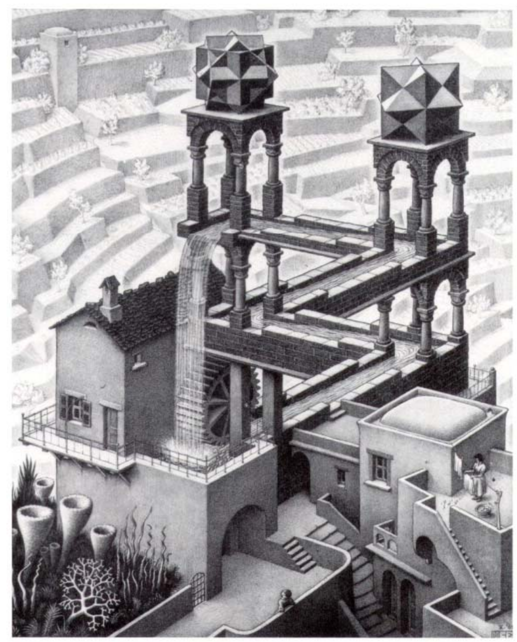
$$D\,(\langle D \rangle) = \begin{cases} \textit{accept}, & \text{if D does not accept } \langle D \rangle \\ \textit{reject}, & \text{if D accepts } \langle D \rangle \end{cases}$$

A contradiction!!! ==> Neither D nor H can exist.

Of Paradoxes & Strange Loops

E.g., Barber’s paradox, Achilles & the Tortoise (Zeno’s paradox)
MC Escher’s paintings



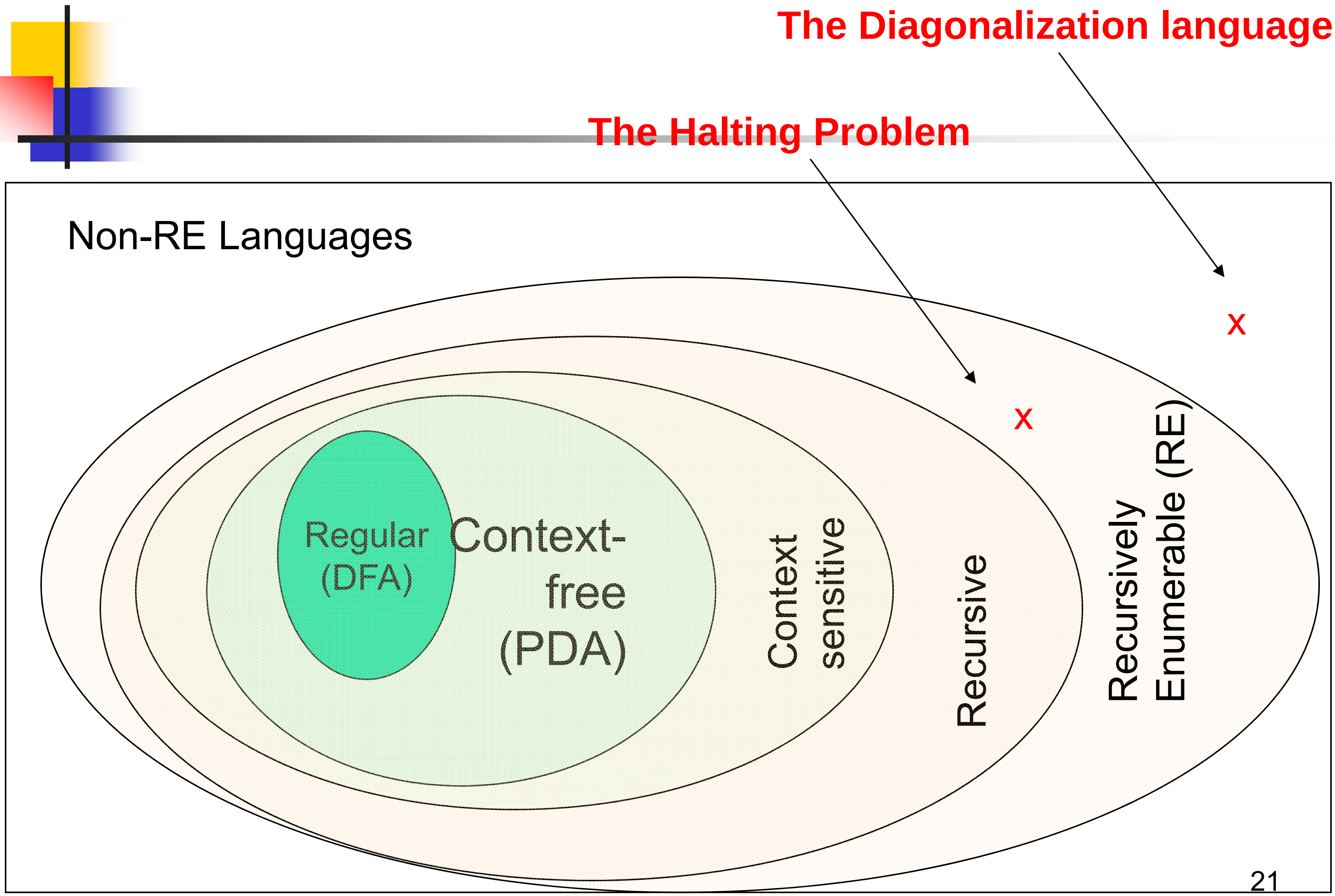


A fun book for further reading:
“Godel, Escher, Bach: An Eternal Golden Braid”
by Douglas Hofstadter (Pulitzer winner, 1980)

The Diagonalization Language

Example of a language that is
<u>not recursive enumerable</u>

(i.e, no TMs exist)


The Diagonalization language
The Halting Problem
Non-RE Languages
x
x
Regular
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A Language about TMs & acceptance

- Let L be the language of all strings <M,w> s.t.:
  1. M is a TM (coded in binary) with input alphabet also binary
  2. w is a binary string
  3. M accepts input w.

Enumerating all binary strings

- Let w be a binary string
- Then 1w $\equiv$ i, where i is some integer
  - E.g., If w=ε, then i=1;
  - If w=0, then i=2;
  - If w=1, then i=3; so on…
- If 1w$\equiv$ i, then call w as the i^{th} word or i^{th} binary string, denoted by w_i.
- ***==> A <u>canonical ordering</u> of all binary strings:***
  - ***{ε, 0, 1, 00, 01, 10, 11, 000, 100, 101, 110, …..}***
  - ***{w_1, w_2, w_3, w_4, …. w_i, … }***

Any TM M can also be binary-coded

- $M = \{ Q, \{0,1\}, \Gamma, \delta, q_0, B, F \}$
  - Map all states, tape symbols and transitions to integers (==>binary strings)
  - $\delta(q_i,X_j) = (q_k,X_l,D_m)$ will be represented as:
    - ==> $0^i 1\ 0^j 1\ 0^k 1\ 0^l 1\ 0^m$
- <u>Result: </u>Each TM can be written down as a long binary string
- ==> Canonical ordering of TMs:
  - $\{M_1, M_2, M_3, M_4, \ldots. M_i, \ldots\}$

The Diagonalization Language

- $L_d = \{ w_i \mid w_i \notin L(M_i) \}$
  - The language of all strings whose corresponding machine does *not* accept itself (i.e., its own code)

(input word w)

j →

(TMs) i ↓		1	2	3	4	…
	1	0	1	0	1	…
	2	1	1	0	0	…
	3	0	1	0	1	…
	4	1	0	0	1	…
	⋮					

diagonal

- Table: $T[i,j] = 1$, if M_i accepts w_j
  $= 0$, otherwise.

- Make a new language called
  $L_d = \{w_i \mid T[i,i] = 0\}$

L_d is not RE (i.e., has no TM)

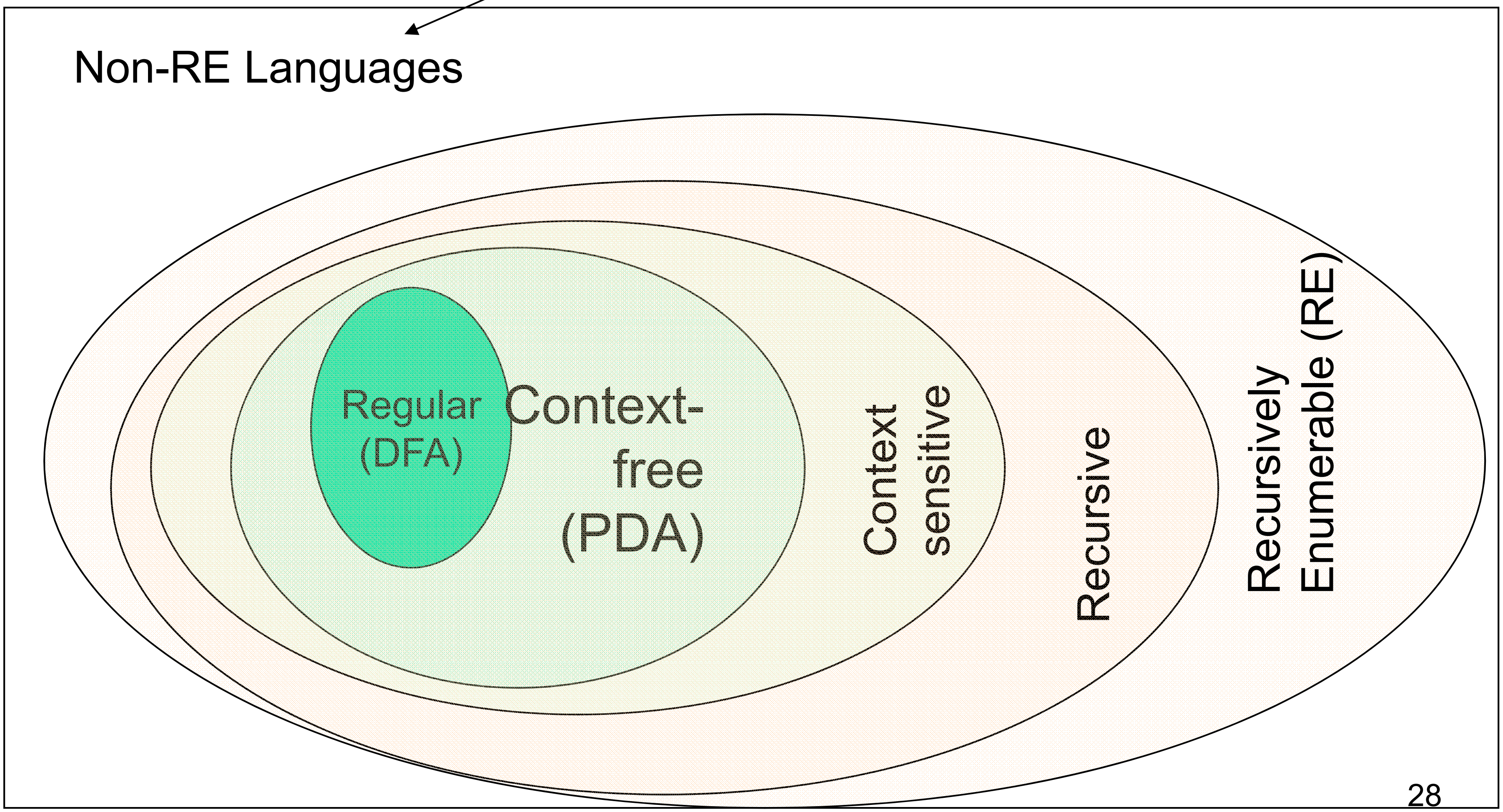
- <u>Proof (by contradiction):</u>
- Let M be the TM for L_d
- ==> M has to be equal to some M_k s.t.
  $L(M_k) = L_d$
- ==> Will w_k belong to $L(M_k)$ or not?
  1. If $w_k \in L(M_k)$ ==> T[k,k]=1 ==> $w_k \notin L_d$
  2. If $w_k \notin L(M_k)$ ==> T[k,k]=0 ==> $w_k \in L_d$
- A contradiction either way!!

Why should there be languages that do not have TMs?

We thought TMs can solve everything!!

Non-RE languages

How come there are languages here?
(e.g., diagonalization language)



One Explanation

There are more languages than TMs

- By pigeon hole principle:
- ==> some languages cannot have TMs

- But how do we show this?
- Need a way to "*count & compare*" two infinite sets (languages and TMs)

How to count elements in a set?

Let A be a set:

- If A is finite ==> counting is trivial
- If A is infinite ==> how do we count?
- And, how do we compare two infinite sets by their size?

Cantor's definition of set "size" for infinite sets (1873 A.D.)

Let N = {1,2,3,…} (all natural numbers)

Let E = {2,4,6,…} (all even numbers)

Q) Which is bigger?

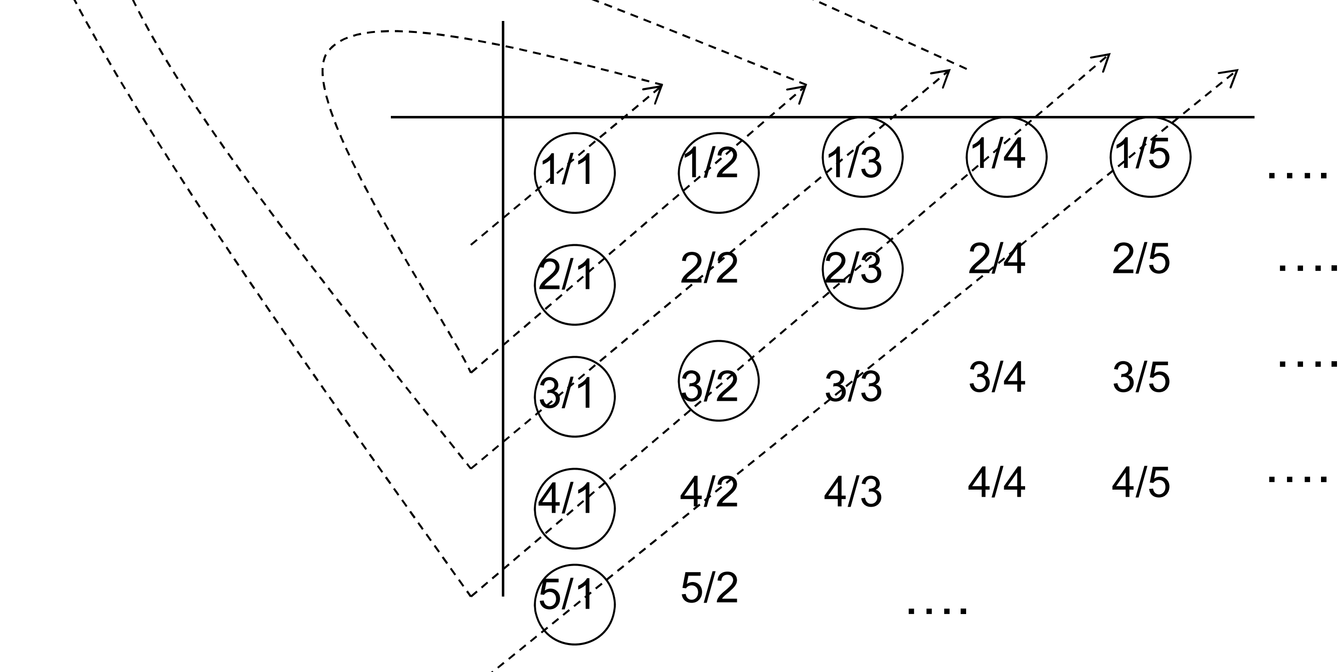
- A) Both sets are of the same size
  - "***Countably infinite***"
  - Proof: Show by one-to-one, onto set correspondence from N ==> E

n	*f(n)*
1	2
2	4
3	6
.	.
.	.

i.e, for every element in N, there is a unique element in E, and vice versa.

Example #2

- Let Q be the set of all rational numbers
- Q = { m/n | for all m,n $\in$ N }
- <u>Claim:</u> Q is also countably infinite; => |Q|=|N|



Really, really big sets!
(even bigger than countably infinite sets)

Uncountable sets

Example:

- Let R be the set of all real numbers
- Claim: R is uncountable

n	*f(n)*
1	3 . $\underline{1}$ 4 1 5 9 …
2	5 . 5 $\underline{5}$ 5 5 5 …
3	0 . 1 2 $\underline{3}$ 4 5 …
4	0 . 5 1 4 $\underline{3}$ 0 …
.	
.	
.	

Build x s.t. x cannot possibly occur in the table

E.g. x = 0 . 2 6 4 4 …

Therefore, some languages cannot have TMs…

- The set of all TMs is countably infinite
- The set of all Languages is uncountable
- ==> There should be some languages without TMs (by PHP)

The problem reduction technique, and reusing other constructions

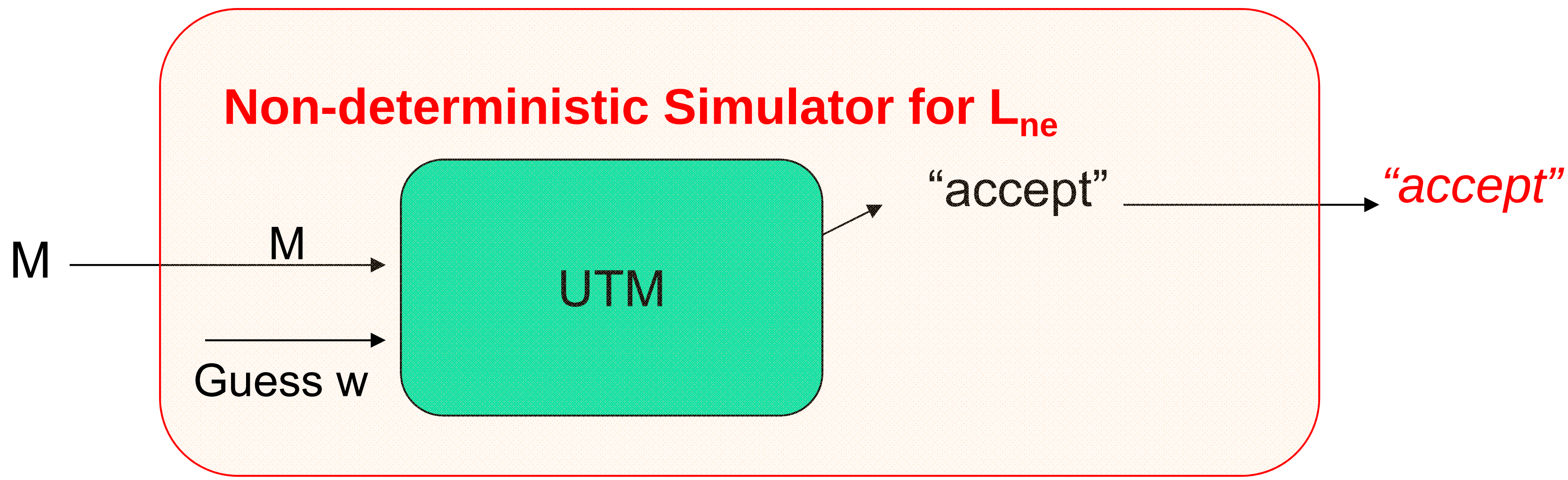
Languages that we know about

- *Language of a Universal TM ("UTM")*
  - $L_u = \{ <M,w> \mid M \text{ accepts } w \}$
  - <u>Result:</u> L_u is in RE but not recursive

- *Diagonalization language*
  - $L_d = \{ w_i \mid M_i \text{ does not accept } w_i \}$
  - <u>Result:</u> L_d is non-RE

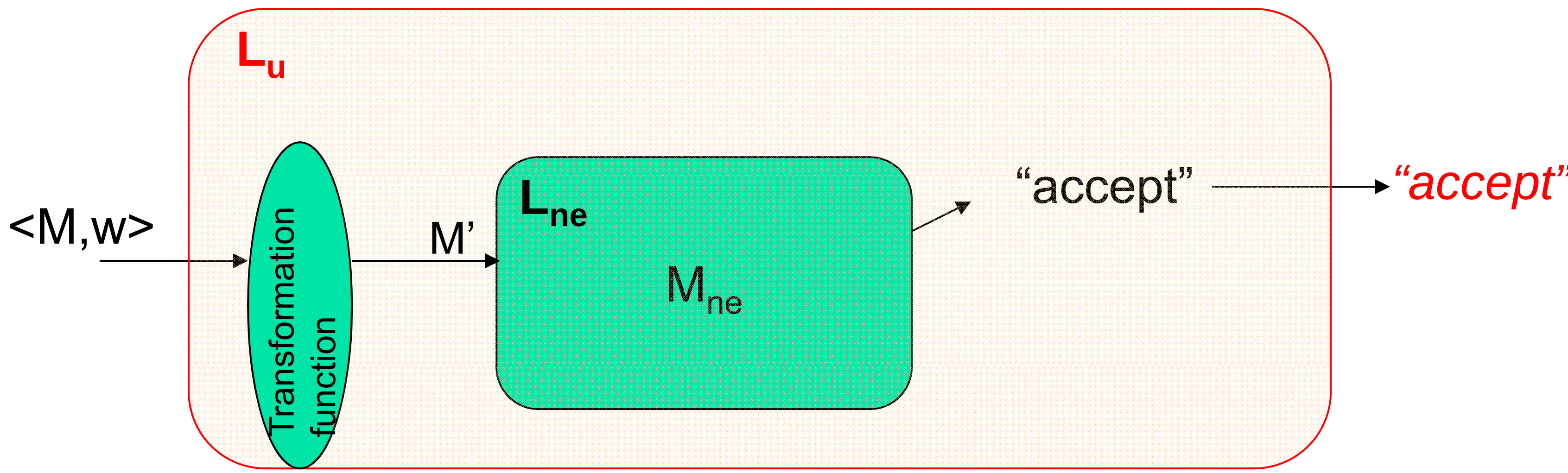
TMs that accept non-empty languages

- $L_{ne} = \{ M \mid L(M) \neq \emptyset \}$
- L_{ne} is RE
- <u>Proof:</u> (build a TM for L_{ne} using UTM)



TMs that accept non-empty languages

- L_{ne} is not recursive
- <u>Proof:</u> ("*Reduce*" L_u to L_{ne})
  - <u>Idea:</u> M accepts w if and only if $L(M') \neq \emptyset$



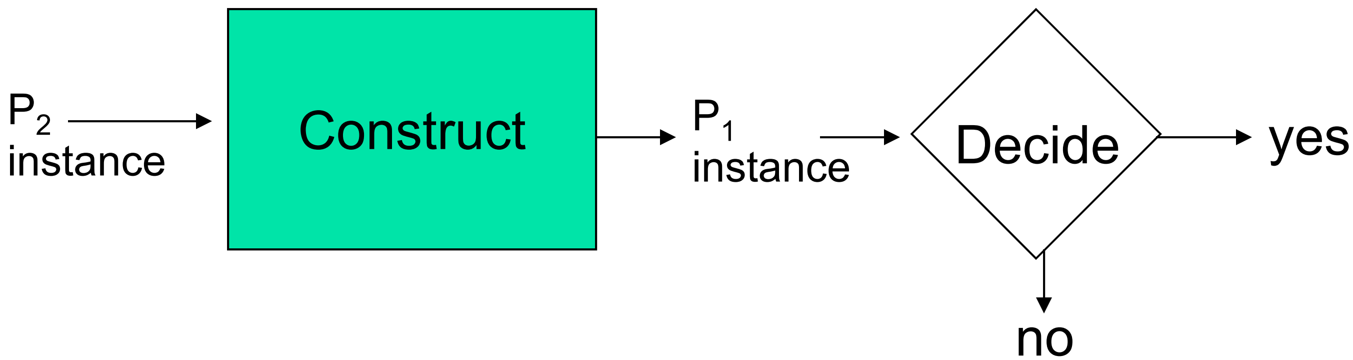
Reducability

- <u>To prove:</u> Problem P_1 is undecidable
- <u>Given/known:</u> Problem P_2 is undecidable
- <u>Reduction idea:</u>
  1. "Reduce" P_2 to P_1:
     - Convert P_2's input instance to P_1's input instance s.t.
       i) P_2 decides only if P_1 decides
  2. Therefore, P_2 is decidable
  3. A contradiction
  4. Therefore, P_1 has to be undecidable

The Reduction Technique

Reduce P_2 to P_1:

__Note:__
not same as
$P_1 ==> P_2$



__Conclusion:__ If we could solve P_1, then we can solve P_2 as well

Summary

- Problems vs. languages
- Decidability
  - Recursive
- Undecidability
  - Recursively Enumerable
  - Not RE
  - Examples of languages
- The diagonalization technique
- Reducability