Influence Models of Cascading Failure and Frequency Oscillation in the Power Grid

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***Abstract*— Power grid is a complex network of connected components interacting with each other. In this paper we focus on the random dynamics of both load cascading failure and the propagation of frequency oscillation in power grids. We effectively identify the vulnerable nodes by the steady state probabilities of *influence model*. Influence model is a type of interconnected Markov chains that could efficiently model the interactive dynamic processes of networked components. We present the procedure to construct both homogenous and heterogeneous influence models using realtime frequency measurements. A low-complexity algorithm is proposed to compute the steady state probability of the Influence model. The computational complexity is dropped from $O(N^6)$ FLOPS to $O(N^4)$ for N-node networks. The computational memory space is reduced by 68%.**

I. Introduction

The United States power grid is a very complex network with over 9000 interconnected power generation components [1], which is shown to be vulnerable to natural or malicious physical events [2]. The frequency oscillations, due to the loss of synchronization on a node or resonant behavior in response to other nodes, are normal behaviors. However such a disturbance propagating through the power grid, together with the catastrophic cascading failure caused by node failure, could bring damage to electronic equipment and produce power outage in a wide area, thus causing substantial economic loss and jeopardizing the national security [3]. There is a prompting need to analyze and manage the risks of such a large scale propagation of node failure and frequency oscillation for power system reliability [4].

The cascading failure and frequency oscillation in power grid could be viewed as the sequence of dependent events of power grid components [4]. Network topology plays an important role in such complex dynamics [4] [7]. It has been widely recognized that network resilience to the cascading failure and frequency oscillation concerns the global structural properties even though failure or frequency disturbance emerges locally. In this paper we consider such dynamics as random events and assess them by the *influence model* [11]. As summarized in [12], it is an interconnected Markov chain that models the random dynamic interactions over a logical structure of influence, other than the physical topology. We assume that each node takes binary state of a Markov model to represent how failure or frequency oscillation propagates among nodes and how nodes of different states influence each other [4]. In this paper we show that the vulnerable nodes crucial to such propagation could be identified by the steady state probability, which can be effectively computed with a computationally efficient algorithm.

Such random dynamics can be assessed by whether or not the initial power grid component could lead to the failure or frequency abnormality of a large number of interconnected components [5]. Note that the results on node cascading failure could be applied to link cascading failure analysis similarly to that of [5] [6], under the duality property that nodes and links of an interconnected structure could be transformed back and forth. [7] proposed a criterion to assess the scale of damage caused by a single initial node. We will show that the criterion in [7] to identify the most vulnerable node is a special case of our model, and our model provides a unified method that could further handle the case of multiple initial nodes.

Influence model has been successfully applied in static and dynamic resource allocations [13], multi-sensor network data fusion [14], and analyzing the conversational human interactions [15]. All these analyses are based on experimental data. However, among the literatures by far applying influence model in power grids, the influence model structure is (at least partially) assumed to be known, instead of being constructed and tested using experimental data. *In this paper we present the scheme to construct influence model based on topology data and power grid frequency measurement data*. The proposed scheme can be applied to any types of network, such as small-world or scale-free. We will test our scheme using Western U.S. power grid topology [22] to analyze the node cascading failure, and frequency measurements of the Frequency Monitoring Network (FNET) [16] to analyze the frequency oscillation.

The remainder of this paper is organized as follows. Section II briefly illustrates the influence model. A low-complexity algorithm is proposed in Section III to compute the steady probabilities in influence model. Section IV presents how to construct homogenous influence model given topology data to analyze cascading failure, while Section V presents how to construct heterogeneous influence model from power grid measurement data to analyze the frequency oscillation. Finally, the conclusions are drawn in Section VI.

II. Influence Model

In this section we use an example of $N = 6$ nodes in [17], as shown in Fig.1, to briefly illustrate how to use influence models with binary states to analyze power grid cascading failure and frequency oscillation.

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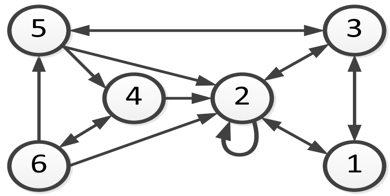

Fig. 1: An example illustrating influence model

A. Homogenous Binary Influence Model

In homogenous binary influence model, each node has binary states: 0 means the node has no failure or normal frequency, 1 shows the node fails or has abnormal frequency.

Such influence model is described by the *topology matrix* $\mathbf{D} = \{d_{ij}\}$. d_{ij} shows the amount of influence from node N_j to N_i. $\mathbf{D}$ is a stochastic matrix with row sum being 1; i.e. the weight on incoming edges to each node sums up to 1. With the topology in Fig. 1, we construct the topology matrix $\mathbf{D}$ in this way: we assume that the amounts of influence are uniformly distributed for all connected node pairs for each node; i.e., it equals the inverse of the node degree. $\mathbf{D}$ is then given by

$$\mathbf{D} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \end{pmatrix}. \quad (1)$$

[11] defines the state vector of dimensions N at time k as $\mathbf{s(k)} = \{s_i(k)\}_{i=1}^N$ with binary value $s_i(k)$ being the state of each node N_i. The conditional probability vector of size $N \times 1$ is $\mathbf{p(k)} = \{p_i(k)\}_{i=1}^N$ with $p_i(k)$ = Pr ($s_i(k+1) = 1 | s_i(k) = 1$). The state evolution is determined by $\mathbf{p(k+1)} = \mathbf{D} \times \mathbf{s(k)}$. The realization of $s_i(k+1)$ is to flip a Bernoulli coin with probability $p_i(k+1)$ for each node. For the steady state probability vector of size $N \times 1$, namely $\pi = \{\pi_i\}_{i=1}^N$, [11] and [17] show that (after a random perturbation), if the directed graph $\Gamma(\mathbf{D}^T)$ is irreducible and aperiodic (ergodic), then we have

$$\lim_{k \to +\infty} \mathbf{D}^k = \mathbf{1} \times \pi^T, \quad (2)$$

$\mathbf{1}$ is an all-one vector, π is the dominant eigenvector of $\mathbf{D}$.

B. Single Node Steady State Probability

[11] shows $\mathbf{E}\Big(\mathbf{s(k)} \mid \mathbf{s(0)}\Big) = \mathbf{D}^k \mathbf{s(0)}$, which implies

$$\lim_{k \to +\infty} \mathbf{E}\Big(\mathbf{s}(k) \mid \mathbf{s}(0)\Big) = \lim_{k \to +\infty} \mathbf{D}^k \mathbf{s(0)} = \mathbf{s}^T(0)\pi. \quad (3)$$

For a single initial node $s_i(0) = 1$ for N_i and $s_j(0) = 0$ $j \neq i$, the probability that all nodes fail (node cascading failure) or abnormal frequency is the steady state probability π_i. We use such a criterion to identify the node most probable to trigger a large scale failure. For the above example, steady state is $\pi = (0.037, 0.053, 0.041, 0.375, 0.206, 0.286)$ for all 6 nodes. N_4 has the highest probability, thus being the one most likely to trigger a cascading node failure or frequency abnormal.

C. Heterogeneous Binary Influence Model

According to [11] the binary states of N_i are vectors $s_i(k) = (0, 1)^T$ (failed or abnormal) or $(1, 0)^T$ (operational or normal). The state of each node is driven by a *local Markov chain* with *state transition probability matrix* $\mathbf{A} = \{\mathbf{A}_{ij}\}$ of size $2N \times 2N$, block partitioned by $\mathbf{A}_{ij} = \{a_{m_j,m_i}\}$ of size 2×2. a_{m_j,m_i} = Pr (N_i takes state m_i at time $k+1 \mid N_j$ has state m_j at k). $\mathbf{A}_{ij}$ represents the different amount of influence from different states of node N_j to N_i.

The heterogeneous influence model is described by two matrices $\mathbf{D}$ (constructed the same as before) and $\mathbf{A}$. Based on the physical meaning of $\mathbf{A}$, it could be constructed if a set of measurement data is available. For example, suppose that we have a frequency measurement data for the 6 power grid components in Fig.1 taking binary value 0(normal frequency) or 1(abnormal frequency). Then $\mathbf{A}_{ij}$ could be estimated by frequency count in this measurement data. A large data set makes frequency count approximate the state transition probability by the law of large numbers. For instance, $f_i(k)$ denotes the frequency measurement of node N_i at time k; then a_{01} in $\mathbf{A}_{ij}$ is the fraction of frequency data pair $(f_i(k) = 0, f_j(k+1) = 1)$ in the whole data set.

[11] defines both $\mathbf{s(k)} = \{s_i(k)^T\}$ and $\mathbf{p(k)}$ of size $2N \times 1$. It also defines the *Influence matrix* $\mathbf{H}$ to be the *Kronecker product* [19]; i.e.,

$$\mathbf{H} \triangleq \mathbf{D}^T \otimes \{\mathbf{A}_{ij}\} \triangleq \begin{pmatrix} d_{11}\mathbf{A}_{11} & \cdots & d_{n1}\mathbf{A}_{1n} \\ \vdots & \ddots & \vdots \\ d_{1n}\mathbf{A}_{n1} & \cdots & d_{nn}\mathbf{A}_{nn} \end{pmatrix}. \quad (4)$$

Then, the state evolution is given by

$$\mathbf{p^T(k+1)} = \mathbf{s^T(k)} \times \mathbf{H}, \quad (5)$$

where $\mathbf{H}$ is the state transition matrix used to derive the single steady state probability π_i.

(5) could be rewritten as $\mathbf{p}_i(k+1) = \sum_{j \in Pa_i} d_{ij} \times \mathbf{s}_j(k) \times \mathbf{A}_{ji}$, showing how d_{ij} weighs the influence coming from all neighboring nodes to N_i; thus we say that $\mathbf{A}_{ij}$ represents the different amount of influence from different states of node N_j to N_i while d_{ij} shows the *general* amount of influence from node N_j to N_i.

D. Joint Steady State Probability

[11] also shows that, for N nodes, there are constant permutation square matrices $\mathbf{M}_2$, $\mathbf{P}_2$ and $\mathbf{T}_2$ with $\mathbf{M}_2 \times \mathbf{T}_2 = \mathbf{I}$, and $\mathbf{E}\Big(\mathbf{y_2^T(k)}\Big) = \mathbf{E}\Big(\mathbf{y_2^T(0)}\Big) \times \mathbf{H}_2^k$, where $\mathbf{y_2^T(k)}$ is the joint state vector of all non-repeat node tuples in lexicographical order, and the joint influence matrix $\mathbf{H}_2$ is given by $\mathbf{H}_2 = \mathbf{M}_2 \times \mathbf{P}_2^T \times (\mathbf{H} \otimes \mathbf{H}) \times \mathbf{P}_2 \times \mathbf{T}_2$.

Suppose that, given measurement data, we could construct $\mathbf{A}$ as $\mathbf{A}_{ii} = \mathbf{I}_2$; for all $i \neq j$, we set

$$\mathbf{A}_{ij} = \begin{pmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}.$$

Then, for the single node steady state probability, we observe that the heterogeneous case still claims that N_4 is the most vulnerable node. The joint steady state probabilities for all

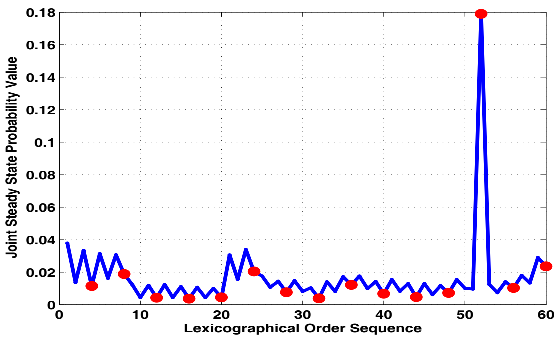

Fig. 2: Joint steady state probabilities in lexicographical order

pairs are computed and plotted in Fig. 2, which shows that the joint probability Pr(node N_4 and N_5 both take state 1 at the same time) takes the peak over all other normalized steady state probabilities. Due to the page limit, for more details about the lexicographical order in this plot and the computation, the readers can refer to [11].

E. Steady State Probability Criterion

Consider a subset of N nodes $\Phi \subsetneqq \{1, , ..., N\}$ and only $s_i(0) = 1$ for $i \in \Phi$; then $\sum_{i\in\Phi} \pi_i$ yields the probability of cascading failure caused by nodes in Φ failing at the same time. Thus in Section II-B node pairs $N_{4,5}$, $N_{4,6}$ and $N_{5,6}$ are highly likely to cause cascading failure if both failed initially. However, this property only holds when nodes in Φ are mutually independent. In fact we observe that only $N_{4,5}$ takes the peak in the joint state probability in Fig.2.

We also performed a node-by-node brute force simulation on cascading failure, according to Section IV-B, on various randomly generated graphs with different sizes, and came up with the conclusion that the node pair with the largest joint steady probability triggers the largest cascading failure with the fastest speed. We hence conclude that the steady state probability could be used as a criterion to identify the most vulnerable node(s):

- For single initial node case, the node with the largest steady state probability is the one most likely to trigger the largest scale of failure asymptotically.
- For multiple initial nodes, those having the largest joint steady probability is the most vulnerable pair to trigger the largest scale of failure fast.

This criterion could help us to identify the most important part of power grid and better design the protective system by taking specific precaution focused on such "vulnerable" components and paying less effort for better reliability.

III. Low-complexity Algorithm for Computing Steady State Probability

In this section, based on the properties of Kronecker matrix product, we present a low-complexity algorithm to efficiently compute the steady state probability. Before we discuss the details, we first state a useful fact. Consider non-singular square matrices A, B, C, D with complying dimensions $(A\otimes B)^T = A^T\otimes B^T$, $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$, and $(A\otimes B)\times(C\otimes D) = (A\times C)\otimes(B\times D)$. For any positive integer k, it is trivial to prove by induction that $(A\otimes B)^k = (A^k)\otimes(B^k)$.

A. Computing $\mathbf{H}$ *and Steady State Probability*

If $\mathbf{D}$ represents a Markov network that is ergodic and the local Markov Chain $\mathbf{A}$ is irreducible, then we have

$$\lim_{k\to+\infty} \mathbf{H}^k = \mathbf{1}^T \times \pi, \tag{6}$$

i.e. $\forall\varepsilon_1,\ \varepsilon_2 > 0$, $\exists$ a large integer $M > 0$ such that the matrix norm $\|\ \mathbf{H}^M - \mathbf{1}\times\pi^T\ \| < \varepsilon_1$, or $\|\ \mathbf{H}^{M+1} - \mathbf{H}^M \| < \varepsilon_2$; thus we write $\mathbf{H}^M$ as $\mathbf{H}^\infty$.

Consider the modal decomposition with eigenvector matrices $\mathbf{U}, \mathbf{V}$ and diagonal eigenvalue matrix $\mathbf{\Lambda}$. We have $\mathbf{D}^T = \mathbf{U}_1\times\mathbf{\Lambda}_1\times\mathbf{V}_1$, $\mathbf{A} = \mathbf{U}_2\times\mathbf{\Lambda}_2\times\mathbf{V}_2$. Then, the property of Kronecker product yields

$$\begin{aligned}\mathbf{H} &= \mathbf{D}^T\otimes\mathbf{A}\\ &= \Big(\mathbf{U}_1\times\mathbf{\Lambda}_1\times\mathbf{V}_1\Big)\otimes\Big(\mathbf{U}_2\times\mathbf{\Lambda}_2\times\mathbf{V}_2\Big)\\ &= \Big(\mathbf{U}_1\otimes\mathbf{U}_2\Big)\times\Big(\mathbf{\Lambda}_1\otimes\mathbf{\Lambda}_2\Big)\times\Big(\mathbf{V}_1\otimes\mathbf{V}_2\Big)\\ &\triangleq \mathbf{U}_3\times\mathbf{\Lambda}_3\times\mathbf{V}_3.\end{aligned} \tag{7}$$

Now we prove that this is a modal decomposition by the following arguments:

- First, $\mathbf{\Lambda}_3 = \mathbf{\Lambda}_1\otimes\mathbf{\Lambda}_2$ is a diagonal matrix.
- Second, note that the eigenvector matrix $\mathbf{V}_i = \mathbf{U}_i^{-1}$ for $i = 1, 2$. Hence we have $\mathbf{U}_3^{-1} = \Big(\mathbf{U}_1\otimes\mathbf{U}_2\Big)^{-1} = \Big(\mathbf{U}_1\Big)^{-1}\otimes\Big(\mathbf{U}_2\Big)^{-1} = \mathbf{V}_1\otimes\mathbf{V}_2 = \mathbf{V}_3$.

If we further require that $\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_N$ for all $\mathbf{\Lambda}_i$, then the eigenvector corresponding to the largest eigenvalue of $\mathbf{H}$ is given by

$$\mathbf{U}_{H1} \triangleq \mathbf{U}_{D1}\otimes\mathbf{U}_{A1}, \tag{8}$$

where $\mathbf{U}_{D1}$ denotes the first row of $\mathbf{U}_1$ and $\mathbf{U}_{A1}$ denotes the first row of $\mathbf{U}_2$, if the eigenvalues are sorted in a descending order. Note that for transition probability matrix, the largest eigenvalue is always 1.

This is also the steady state probability as the M-th power of such a matrix only changes its eigenvalue. We could prove this property using the properties of Kronecker product:

$$\begin{aligned}\mathbf{H}^\infty &= \Big(\mathbf{D}^T\Big)^\infty\otimes\Big(\mathbf{A}\Big)^\infty\\ &= \Big(\mathbf{U}_3\times\mathbf{\Lambda}_3\times\mathbf{U}_3^{-1}\Big)^\infty,\end{aligned}$$

$$\begin{aligned}&= \mathbf{U}_3\times\mathbf{\Lambda}_3\times\Big(\mathbf{U}_3^{-1}\times\mathbf{U}_3\Big)\times\mathbf{\Lambda}_3\times\mathbf{U}_3^{-1}\\ &\quad\underbrace{\times\cdots\times\mathbf{U}_3\times\mathbf{\Lambda}_3\times\mathbf{U}_3^{-1}}_{\text{M factors in total}}\\ &= \mathbf{U}_3\times\Big(\mathbf{\Lambda}_3\Big)^\infty\times\mathbf{V}_3.\end{aligned} \tag{9}$$

B. Computing $\mathbf{H}_2$ and Joint Steady State Probability

It is obvious that the permutation matrix $\mathbf{P}_2$ has the unitary property $\mathbf{P}_2^T = \mathbf{P}_2^{-1}$. Although $\mathbf{T}_2\mathbf{M}_2 \neq \mathbf{I}$, its row-column permutation $\mathbf{P}_2(\mathbf{T}_2\mathbf{M}_2)\mathbf{P}_2^T = \mathbf{I}$. Similarly we have

$$\begin{aligned}\mathbf{H}_2^\infty &= \Big(\mathbf{M}_2 \times \mathbf{P}_2^T \times (\mathbf{H}\otimes\mathbf{H}) \times \mathbf{P}_2 \times \mathbf{T}_2\Big)^\infty, \\ &= \mathbf{M}_2\mathbf{P}_2^T(\mathbf{H}\otimes\mathbf{H})\Big[\mathbf{P}_2\Big(\mathbf{T}_2\times\mathbf{M}_2\Big)\mathbf{P}_2^T\Big](\mathbf{H}\otimes\mathbf{H})\mathbf{P}_2\mathbf{T}_2 \\ &\qquad \underbrace{\times\cdots\times\mathbf{M}_2\mathbf{P}_2^T(\mathbf{H}\otimes\mathbf{H})\mathbf{P}_2\mathbf{T}_2}_{\text{M factors in total}} \\ &= \mathbf{M}_2 \times \mathbf{P}_2^T \times \Big(\mathbf{H}\otimes\mathbf{H}\Big)^\infty \times \mathbf{P}_2 \times \mathbf{T}_2, \end{aligned} \tag{10}$$

When nodes size N is fixed, $\mathbf{M}_2$, $\mathbf{T}_2$ and $\mathbf{P}_2$ are constant. $\mathbf{H}_2^\infty$ is then the fixed row-column permutation of $(\mathbf{H}\otimes\mathbf{H})^\infty$. We could prove that their eigenvectors are permuted in the same manner and they share the same eigenvalues.

We apply the modal decomposition in (9) and have

$$\begin{aligned}\Big(\mathbf{H}\otimes\mathbf{H}\Big)^\infty &= \mathbf{H}^\infty \otimes \mathbf{H}^\infty \\ &= \Big(\mathbf{U}_3(\mathbf{\Lambda}_3)^\infty\mathbf{V}_3\Big) \otimes \Big(\mathbf{U}_3(\mathbf{\Lambda}_3)^\infty\mathbf{V}_3\Big) \\ &= \Big(\mathbf{U}_3\otimes\mathbf{U}_3\Big)\Big(\mathbf{\Lambda}_3^\infty\otimes\mathbf{\Lambda}_3^\infty\Big)\Big(\mathbf{V}_3\otimes\mathbf{V}_3\Big). \end{aligned} \tag{11}$$

(11) is also a modal decomposition, with $\mathbf{\Lambda}_3^\infty\otimes\mathbf{\Lambda}_3^\infty$ being the diagonal matrix, $\mathbf{U}_3\otimes\mathbf{U}_3$ the left eigenvector matrix and $\mathbf{U}_{H1}\otimes\mathbf{U}_{H1}$ the eigenvector corresponding to the largest eigenvalue. The joint steady state probability is simply the fixed permutation, $\mathbf{U}_{H1}\otimes\mathbf{U}_{H1}$ which by (9) is

$$\begin{aligned}\mathbf{U}_{H_21} &= permute\Big(\mathbf{U}_{H1}\otimes\mathbf{U}_{H1}\Big) \\ &= permute\Big(\mathbf{U}_{D1}\otimes\mathbf{U}_{A1}\otimes\mathbf{U}_{D1}\otimes\mathbf{U}_{A1}\Big). \end{aligned} \tag{12}$$

The heterogeneous case has a similar low-complexity procedure with the *Khatri-Rao* and *Tracy-Singh* matrix product [19]. Details are left to our longer version due to the limited space.

C. Complexity Analysis

(12) provides an easy way to compute the joint steady state probability avoiding the manipulation of very large sparse matrices for large N. The complexity of this method is to compute the dominant eigenvector of an $N\times N$ matrix and a 2×2 matrix, and to figure out the permutation matrix $\mathbf{M}_2\times\mathbf{P}_2$ which is a constant sparse matrix of size $2N(N-1)\times 4N^2$ that only differs on N; hence, the FLOPS (number of floating number operations, including multiplication and add) needed is approximately $O(128\times N^4)$ for permutation and another $O(8\times N^3)$ to compute the steady state probability vector, much less than $O(72\times N^6 + 196\times N^4)$ by directly computing $\mathbf{H}_2$ matrix. The total computational complexity is dropped from $O(N^6)$ FLOPS to $O(N^4)$, with the size of ROM (worst case) saved by 68.36%.

IV. Construction of Topology Matrix $\mathbf{D}$

In this section we construct $\mathbf{D}$ from the Western U.S. power grid topology [22] using the topology based model in [7] [10], similarly to SectionII-A, to analyze node cascading failure.

A. Topology Based Model of Cascading Failure

Node degree and other topology information are critical to the dynamics of propagation. Topology based models view the power grid as a graph with N nodes (power generators) $\{N_i\}_{i=1}^N$, interconnected by undirected edges (transmission lines). The node degree k_i of node N_i measures how N_i is connected to its neighboring nodes Pa_i (set of nodes that are directly connected to N_i in the topology). [8] considered node failure and defined cascading model (initial load on each node and load re-distribution rule when one node fails) to be proportional to the single node degree k_i. [9] studied link failure and defined the load on each link to be proportional to k_ik_j, by taking two connected nodes into consideration. [10] further developed this idea such that the node as well as all its neighboring nodes are taken into consideration to define the load. [7] used the same model and showed that it could effectively identify the most vulnerable node.

In SectionII-C, (5) is rewritten in $\mathbf{p}_i(k+1) = \sum_{j\in Pa_i} d_{ij}\times \mathbf{s}_j(k)\times\mathbf{A}_{ji}$ showing how d_{ij} weighs the influence coming from all neighboring nodes to N_i. Thus, the same as [10], d_{ij} should also be proportional to both the node and its neighbors. Given a topology for N nodes and degree k_i for each node N_i, similarly to that in [7] [10], we define $\widehat{\mathbf{D}} = \{\hat{d}_{ij}\}$ as

$$\hat{d}_{ij} = k_i \times (\frac{k_j}{\sum_{n\in Pa_j} k_n}) \triangleq k_i\times\omega_j. \tag{13}$$

Then we normalize it to make $\mathbf{D}$ be a right stochastic matrix (row summed to 1), and randomly perturb it if necessary, thus making $\Gamma(\mathbf{D}^T)$ ergodic. We can prove that [17] after these steps, $\mathbf{D}^T$ and $\widehat{\mathbf{D}}^T$ share the same eigenvectors and differ only in eigenvalues by a positive scalar $\sigma(\widehat{\mathbf{D}}^T) = \beta\times\sigma(\mathbf{D}^T)$.

Simulation results in [7] showed that $max\omega_j$ is quite effective in identifying the most vulnerable node. Note that the column sum of $\widehat{\mathbf{D}}$ is $\sum_j d_{ij} = (\sum_i k_i)\times\omega_j$; hence $max\omega_j$ is equivalent to the maximum column sum (matrix 1-norm) of $\widehat{\mathbf{D}}$, or maximum row sum (matrix ∞-norm) of $\widehat{\mathbf{D}}^T$. Now we prove the following proposition. A direct corollary is that our criterion using influence model is the same as the criterion proposed in [7].

Proposition 1: For $\widehat{\mathbf{D}}^T$, the maximum row sum and the largest element in dominant eigenvector have the same location.

Proof: For $\widehat{\mathbf{D}}^T$, denote its row by $\mathbf{c}_i^T$ with row sum $\hat{\omega}_j$, dominant right eigenvector $\mathbf{v} = \{v_i\}$. For binary vector $\bar{v}$ we use a unique 1 element to indicate the location of v_{max}, namely the largest elements in $\mathbf{v}$. Note the largest eigenvalue of stochastic matrix $\mathbf{D}^T$ is always 1.

Then we have $\widehat{\mathbf{D}}^T\times\mathbf{v} = \beta\times\mathbf{v}$ and

$$\begin{aligned}\bar{v}^T\times(\widehat{\mathbf{D}}^T\times\mathbf{v}) &= \beta\times\bar{v}^T\times\mathbf{v} = \beta v_{max} \\ (\mathbf{v}^T\times\widehat{\mathbf{D}})\times\bar{v} &= \mathbf{v}^T\times(\mathbf{c}_1,\mathbf{c}_2,...,\mathbf{c}_N)\times\bar{v} = \beta v_{max}.\end{aligned}$$

Suppose $\max(\hat{\omega}_j) = \mathbf{1}^T\times\widehat{\mathbf{D}}\times\bar{u}$ where $\bar{u}$ indicates the location of the maximum column sum in $\widehat{\mathbf{D}}$. We can prove it by contradiction. Suppose $\bar{u}\neq\bar{v}$ and $\widehat{\mathbf{D}}\bar{u} = \mathbf{c}_u$, $\widehat{\mathbf{D}}\bar{v} = \mathbf{c}_v$. Note that all elements here are nonnegative. Then we should have $\mathbf{v}^T\mathbf{c}_u > \mathbf{v}^T\mathbf{c}_v$. This contradicts with what we already have,

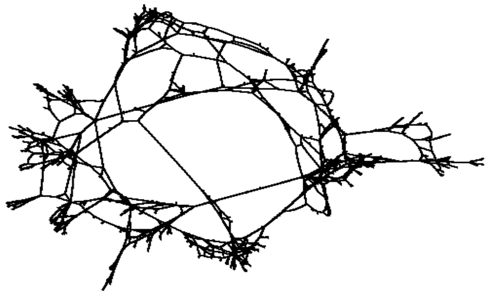

Fig. 3: The Western North American Power Grid Network Topology

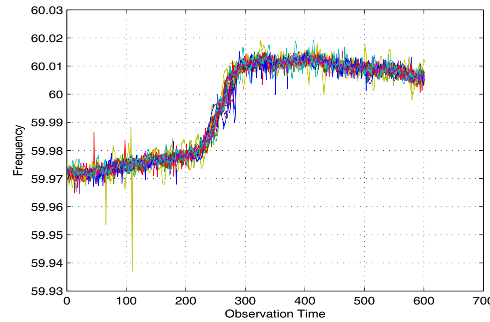

Fig. 4: the First Minute frequency measurement data of all 87 nodes

namely $\mathbf{v}^T\mathbf{c}_v = \mathbf{v}^T \times (\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_N) \times \bar{v} = \beta v_{max} = \max \mathbf{v}^T$. ■

Now we have proved $\bar{u} = \bar{v}$. This implies that the criterion used in [7] be the same one we used in the influence model. Although [7] provided a simpler criterion for single initial node using homogenous model, our criterion could handle the case of multiple initial nodes using more accurate heterogeneous model, as illustrated at end of SectionII-C.

Furthermore, if we define $p_i(k)$ = Pr ($s_i(k) = 1$) instead of using (5), we have $\mathbf{p}^T(k+1) = \mathbf{p}^T(k) \times \mathbf{H}$. In this way we reformulate the influence model into *matrix population model* [18], and the *Perron-Frobenius theorem* [19] [20] shows that the dominant eigenvector represents the asymptotic geometric rate of increasing and asymptotically the largest $||\mathbf{p}||$ (the largest scale of failure possible to happen) is achieved by initially using $\mathbf{s}(0) = \bar{v}$. In this way we proved the criterion that we used in Section II-E is correct using the tool of matrix population model.

B. Numerical Simulation Result

The Western U.S. power grid network [21] is widely used to test topology related methods in power grid analysis. The topology is plotted in Fig.4. The data contains a topology of $N = 4941$ nodes and 6594 edges, and the corresponding influence model is a homogenous binary type. Both criteria obtained from influence model and proposed in [7] identify node N_{3469} as the most vulnerable one to trigger the largest scale of cascading failure. Furthermore, influence model provides its steady state probability being 0.0102, significantly larger than others. Although the low-complexity algorithm still takes over a month on an Intel Core i750 workstation with 5 GFLOPS on average to compute the joint steady state probability for all possible node pairs, a local search focused on N_{3469} shows that N_{2059} is most likely to fail with N_{3469} at the same time, while the single node probability of N_{2059} only ranks the 2793^{rd} largest in all 4941 nodes. This observation is the same as the example in Section II that, although N_5 does not have the second largest single node steady state probability, the pair achieves the maximum in the joint probability.

The cascading simulator in [7] did not allow failed node to recover and runs in a deterministic way, while the influence model allows such a recovery and runs in a stochastic way. 500 independent brutal force cascading simulations on the influence model show that N_{3469} triggers the largest scale of cascading failure, and N_{2059} is on average the 17.3^{th} node first visiting the failed state due to the initial failure of N_{3469}, with average returning time 97.81. It is consistent with result on ergodic Markov chain compared with the theoretical value $1/\pi \approx 98.04$. This observation may lead to deeper insights on the expected returning time and stopping time in the further research of influence model.

V. Construction of Local State Transition Probability Matrix $\mathbf{A}$

Previously we have shown the application of homogenous influence model in cascading failure analysis. Essentially, influence model uses a structure which is based on (but different from) the physical structure to model the interactions and dynamics of interconnected power grid components.

In this section, we apply the heterogeneous influence model and use the states Normally Operational (0) and Abnormally Operational (1) to represent such influence between components on frequency, which causes the propagation of frequency oscillation during a generator trip event in Florida on Dec 2010, with the first 1 minute of frequency data plotted in Fig.4. A similar event could be visualized on-line at http://www.youtube.com/watch?v=bdBB4byrZ6U, showing how the frequency disturbance propagated and oscillated over the whole nation's power grid.

A. Constructing $\mathbf{A}$ from Measurement Data

FNET [16] provides real time frequency measurements of power grid. Frequency data is collected from $N = 87$ Frequency Disturbance Recorders (FDRs) located across North America. The data sampling period is 100ms and the samples are quantized into 7 levels for frequency within $(59.93Hz, 60.03Hz)$. To simplify the analysis, we define binary state on each node to be state 0 when the frequency is within $(59.99Hz, 60.01Hz)$ (low frequency deviation) and state 1 otherwise.

$\mathbf{A}_{ij}$ could be estimated by frequency count in the measurement data as in SectionII-C, $\mathbf{D}$ is constructed from a logical structure as in Section IV, based on the logical topology in [23] plotted in Fig.5 which stems from the skeleton of a Bayesian network structure [24] learned from the same frequency measurements (with edge directions removed).

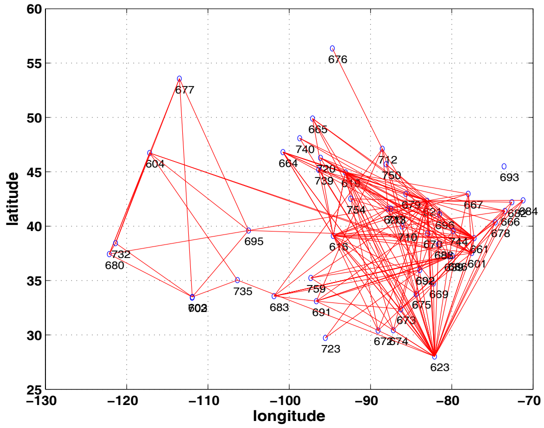

Fig. 5: the topology of logical structure learned from frequency data

B. Numerical Simulation Result

The steady state probabilities are computed after $\mathbf{A}$ and $\mathbf{D}$ are constructed from the logical topology Fig.5. It reveals that node 623, a node with a high degree in the logical topology, is the node most likely to trigger a large scale frequency oscillation. We observe that the node is close to Florida which is the source of this event. Node pair $(616, 667)$ achieves the maximum joint probability. So far, the heterogeneous influence model constructed from the measurement data identifies these nodes to be critical in frequency oscillation.

We further observe that all the three nodes identified by influence model are all among the nodes having the highest degrees in the logical structure. On the one hand, such a logical structure is constructed based on cause-effect probability associations of frequency data among nodes. Hence nodes with higher degree show more importance in the causality of frequency oscillation. On the other hand, influence model identifies the nodes causing the highest amount of influence in the propagation of frequency oscillation. This consistent results show that, even when the physical topology is not available, we could still identify the vulnerable nodes with more importance and stronger influence in the propagation of frequency fluctuation using heterogeneous influence model and measurement data.

VI. Conclusion

In this paper we have presented the application of influence model in power grid analysis. We have proved that the steady state probability, as a criterion of vulnerability, could identify the set of initial nodes most likely to cause a large scale cascading failure or having a strong influence in the propagation of frequency oscillation. We have also proved that the criterion used in [7] is a special case of the homogenous influence model. A low-complexity algorithm has been proposed to compute such steady state probabilities with over 68% reduction in ROM space and computational complexity dropping from $O(N^6)$ FLOPS to $O(N^4)$. Finally, we have presented the scheme to construct the homogenous influence model based on the Western U.S. power grid topology to analyze the cascading failure, and to construct the heterogeneous model based on the FNET frequency measurement data to analyze the frequency oscillation.

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