## Disjoint Sets

Keep keys in disjoint sets

Find set containing key

Union two sets

Application: determine connected components of an undirected graph

```
make each vertex a set
foreach edge
   union sets containing vertices of edge
```

## Representation

A ________________ is a data structure S = $\{S_1, .., S_k\}$, or a collection of disjoint dynamic sets.

Each set has a ___________________ element, which never changes unless unioned with another set.

## Operations

x = pointer to an object containing some key

Make-Set(x)
    Create new set $S_x$ with one member x
    Representative of $S_x$ is x
    Disjoint set = Disjoint set + $S_x$

Union(x,y)
    $S_x$ = set containing x
    $S_y$ = set containing y
    $S_u = S_x \cup S_y$
    rep($S_u$) = rep($S_x$) or rep($S_y$)    ; or any other object in $S_u$
    Disjoint set = Disjoint set $-S_x - S_y + S_u$

Find-Set(x)
    $S_x$ = set containing x
    return rep($S_x$)

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## Application

Finding the connected components of a graph.

Connected-Components(Graph)
    foreach v in vertices(Graph)
        Make-Set(v)
    foreach e in edges(Graph)
        (u,v) = e
        if Find-Set(u) $\neq$ Find-Set(v)
        then Union(u,v)

Same-Component(u,v)
    if Find-Set(u) = Find-Set(v)

then return True
else return False

Example

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**Linked-List Representation**

Use linked list to represent set of objects.

Each object contains a pointer to the rep, the key, and a pointer to next.

## Operations

Make-Set(x) ; $O(1)$
rep(x) = x
next(x) = NIL

Find-Set(x) ; $O(1)$
return rep(x)

Union(x, y) ; $\Theta$(size of x)
foreach object in rep(x)
insert object into y
rep(object) = rep(y)
remove x

---

**Analysis**

The worst case scenario is:

- Make-Set($x_1$)
- ...
- Make-Set($x_n$) $\{\{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_n\}\}$
- Union($x_1$, $x_2$) $\{\{x_1 \rightarrow x_2\}, \{x_3\}, \ldots, \{x_n\}\}$
- Union($x_2$, $x_3$) $\{\{x_1 \rightarrow x_2 \rightarrow x_3\}, \ldots, \{x_n\}\}$
- ...
- Union($x_{q-1}$, $x_q$) $\{\{x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \rightarrow x_n\}$

n = #Make-Set operations

m = #Make-Set, Union, and Find-Set operations

m = n + (q - 1) operations

$$\begin{aligned} \text{T(m)} &= \Theta(n) \ + \ \Sigma_{i=1}^{q-1} i \\ &= \Theta(n \ + \ q^2) \\ \text{n} = \Theta(m) &\text{ and q} = \Theta(m) \end{aligned}$$

Therefore, T(m) = $\Theta(m^2)$ and the amortized cost is $\Theta(m)$ per operation.

Can we do better?

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## Weighted-Union Heuristic

**Idea:** Keep track of the number of objects in a set (length of list). Append shorter list to longer list.

# Theorem 22.1

A sequence of m operations, n of which are Make-Set operations, takes O(m + n lg n) time.

**Proof:** Since we only change rep(x) for objects in the shorter list for each Union, and lists start at length=1, then each Union at least doubles the size of x's list. Thus, we can do at most $\lceil lgn \rceil$ Unions that require rep(x) changes, and there are n objects.

As a result, there are a total of ____________ changes.

If we add the O(1) costs for the O(m) Make-Set and Find-Set operations, we get ________________

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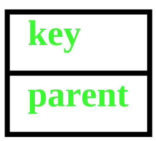
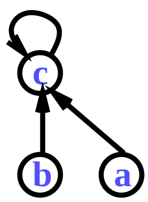
## Disjoint Sets as Forest of Trees

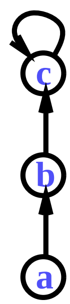
**Idea:** Represent disjoint sets as a forest of trees.

**Object:**




**Example: Sc = {c, b, a}**




Representatives are roots

x = parent(x)

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## Operations

Make-Set(x)

parent(x) = x

FindSet(x)

```
    Follow parent pointers from x to root
    return root

Union(x, y)
    parent(x) = FindSet(y)
```

Performance same as linked lists, but can do better.

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## Union by Rank

In Union, have parent of shallower tree point to other tree.

Maintain rank(x) as an upper bound on the depth of the tree rooted at x.

```
Make-Set(x)
    parent(x) = x
    rank(x) = 0

Union(x, y)
    repx = FindSet(x)
    repy = FindSet(y)
    if rank(repx) > rank(repy)
    then parent(repy) = repx
    else parent(repx) = repy
        if rank(repx) = rank(repy)
        then rank(repy) = rank(repy) + 1
```

Can we do even better?

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### Path Compression

While looking for rep(x) by traversing parent pointers, set each one to the resulting rep(x).

Click on mouse to advance to next frame.

**Note:** Since rank is an ______________ on tree height, path compression need not change ranks.

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### Pseudocode

FindSet(x)
  if x $\neq$ parent(x) ; Two-Pass Method
  then parent(x) = FindSet(parent(x))
  return parent(x)

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### Analysis:

## Union by Rank Only:

$$\Theta(mlgn)$$

m = #operations

n ($\leq$ m) = #MakeSet operations in m

## Path Compression Only:

$\Theta(f \log_{(1+f/n)} n)$ if f $\geq$ n

$\Theta(n + f \lg n)$ if f < n

n = #MakeSet operations

There are $\leq$ n-1 Unions

f = #FindSet operations

**Analysis**

# Union by Rank and Path Compression:

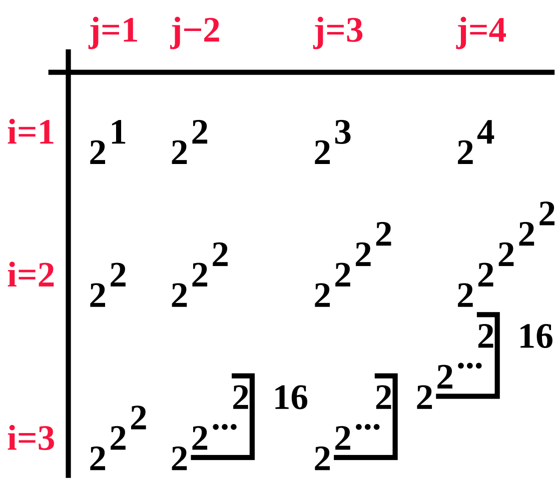
O(m * $\alpha$(m,n)) worst case running time

$\alpha$(m,n) is inverse of Ackermann's function A(i,j)

$\alpha$(m,n) = min{i $\geq$ 1 | A(i, $\lfloor \frac{m}{n} \rfloor$) > lg n}

# Ackermann's Function A(i,j)

- A(1, j) = $2^j$ for j $\geq$ 1
- A(i, 1) = A(i-1, 2) for i $\geq$ 2
- A(i, j) = A(i-1, A(i, j-1)) for i, j $\geq$ 2




**Note:** A(i,j) is strictly increasing and $\lfloor \frac{m}{n} \rfloor \geq 1$ since m $\geq$ n.

Therefore A(4, $\lfloor \frac{m}{n} \rfloor$) $\geq$ A(4,1) = A(3,2)

A(3,2) = 2 raised to the power 2 16 times >> $10^{80}$

$10^{80}$ = the number of atoms in the observable universe

$\alpha(m, n) = 4$ for practical uses since lg n is typically less than $10^{80}$

Thus, T(m) = O(m).
O(1) amortized cost per operation

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## Applications