Graph Algorithms

Graphs are important data structures.

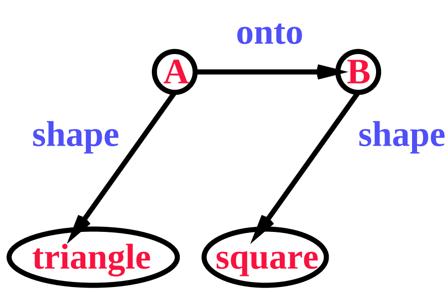
Graphs can express arbitrary relationships between objects.

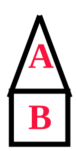
Simple Graphs

- Vertices
- Edges (—, ⟶)

Labelled Graphs

- Vertices and vertex labels
- Edges and edge labels





For now, simple graphs.

A graph G consists of a set of vertices V and a set of edges E such that $(u,v) \in E \rightarrow u, v \in V$ and u is connected to v with an edge.

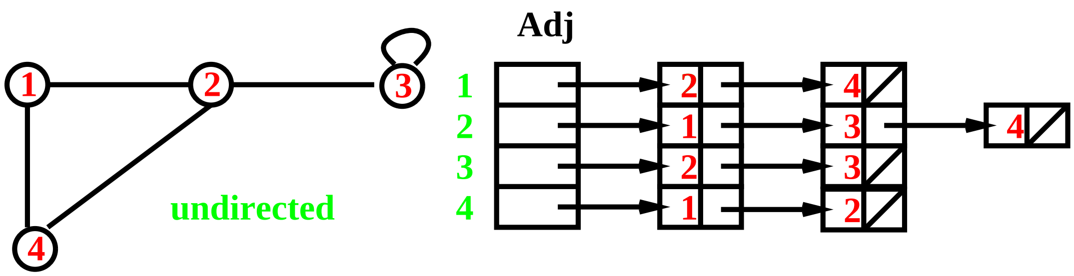
Directed edge: u → v
Undirected edge: u — v

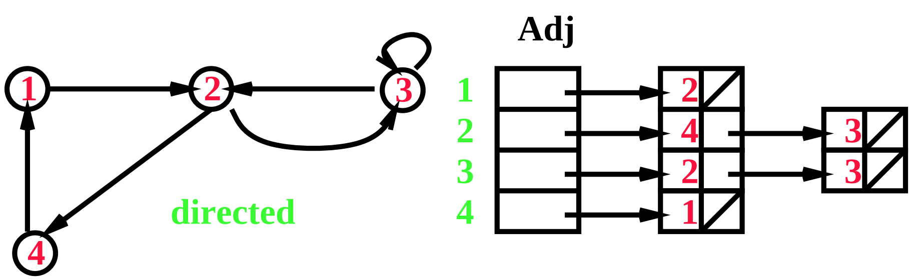
Representation

1. Adjacency lists

2. Adjacency matrix

1. Adjacency list
Array Adj of |V| lists, O(E)
Adj[u] is a pointer to a list of vertices v such that (u,v) ∈ Edges
Memory ________, Lookup ______________





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2. Adjacency matrix

Matrix A |V| x |V|, where

$$A[i,j] = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Memory $\Theta(V^2)$, Lookup O(1)

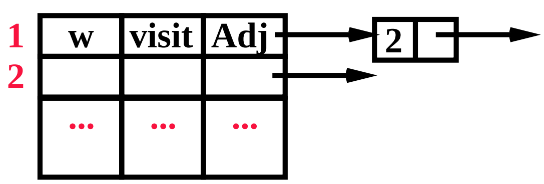
	1	2	3	4
1	0	1	0	1
2	1	0	1	1
3	0	1	1	0
4	1	1	0	0

undirected

	1	2	3	4
1	0	1	0	0
2	0	0	1	1
3	0	1	1	0
4	1	0	0	0

directed

Both allow added satellite data easily



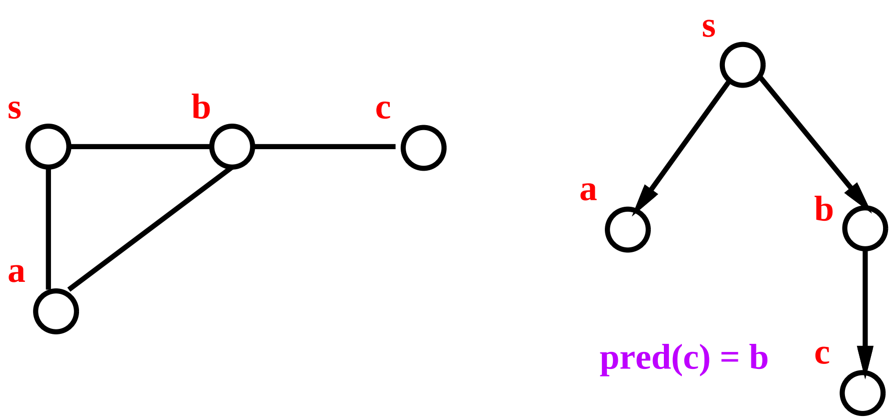
Adjacency list better for sparse graphs

Traversing Graphs

Search for paths satisfying various constraints (e.g., shortest path)
Visit some set of vertices (e.g., tours)
Search for subgraphs (e.g., graph matching (isomorphisms))
Techniques:

1. Breadth-First Search (BFS)
2. Depth-First Search (DFS)

Breadth-First Search (BFS)



Input Graph   Breadth−First Tree

Breadth-first search produces breadth-first tree.
Path from s to x in BF tree is the shortest path in terms of the number of edges.
BFS: Given graph G = (V,E) and source s

- Visit every vertex reachable from s in one edge that has not already been visited
- Visit every vertex reachable from s in two edges that has not already been visited
- ...

BFS Data Structures

A node in a BF tree represents a vertex

pred
vertex
distance
visited

Use a queue to remember frontier of search

Note: Cormen et al.'s BFS algorithm uses *color* instead of *visited*.

- Unvisited vertex: white
- Discovered vertex: gray
- Visited vertex: black

Pseudocode

```
BFS(G,s)
    foreach v in (V - {s})          ; initialize
        visited(v) = False
        pred(v) = NIL
        distance(v) = ∞
    visited(s) = True               ; visit start vertex
    distance(s) = 0
    pred(s) = NIL
    Enqueue(Q, s)
    while not QueueEmpty(Q)
        u = DeQueue(Q)
        foreach v in Adj[u]
```

```
if not visited(v)
then visited(v) = True
    distance(v) = distance(u) + 1
    pred(v) = u
    Enqueue(Q, v)
```

Examples

Analysis

______ Enqueue / Dequeue operations

each vertex is processed only once

______ total time scanning adjacency list

_____

_________ BFS running time worst case

Properties

$\delta(s, v)$ = shortest-path distance from s to v

Theorem 23.4

G = (V, E) directed or undirected

BFS(G, s), s in V

Upon termination of BFS, every vertex v in V reachable from s has distance(v) = $\delta(s, v)$

For vertex v $\neq$ s reachable from s, one shortest path from s to v is the shortest path from s to pred(v) followed by edge (pred(v), v)

Proof: by induction on distance from s

```
Print-Path(G, s, v)                    ; O(V)
    if v = s
    then print s
    else if pred(v) = NIL
        then "no path"
        else Print-Path(G, s, pred(v))
            print v
```

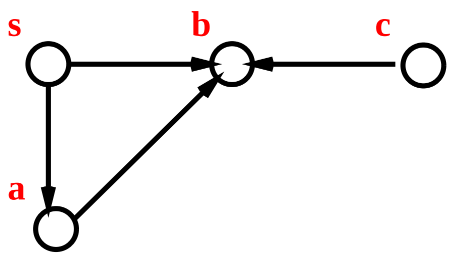
Predecessor Subgraph

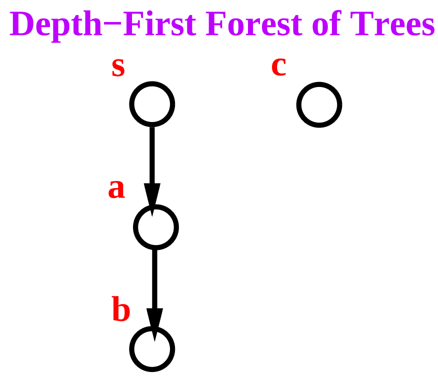
$G_{pred} = (V_{pred}, E_{pred})$ is a **predecessor subgraph** of G if

$$V_{pred} = \{v \in V \mid pred(v) \neq NIL\} \cup \{s\}$$
$$E_{pred} = \{(pred(v), v) \in E \mid v \in V_{pred} - \{s\}\}$$

Lemma 23.5 The **pred** tree generated by BFS results in a predecessor subgraph G_{pred} which defines a BF tree.

Depth-First Search





Vertex in DFF of T

pred
visited
discover
finish

Note again, Cormen et al. use *color* instead of *visited*.

Discover is the time when vertex first visited.

Finish is the time when all vertices reachable from this vertex have been visited.

DFS

1. Given G
2. Pick an unvisited vertex v, remember the rest
3. Recurse on vertices adjacent to v

DFS(G)

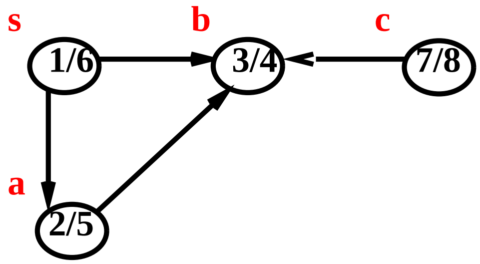
foreach v in V

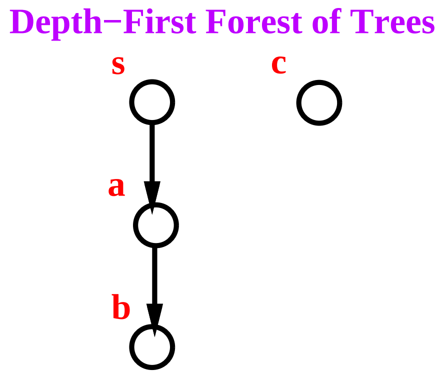
```
        visited(v) = False
        pred(v) = NIL
    time = 0
    foreach u in V
        if not visited(u)
        then Visit(u)

Visit(u)
    visited(u) = True
    time = time + 1
    discover(u) = time
    foreach v in Adj[u]                ; Θ(E)
        if not visited(v)
        then pred(v) = u
            Visit(v)
    time = time + 1
    finished(u) = time
```

The total running time for DFS is $\Theta(V + E)$

Examples

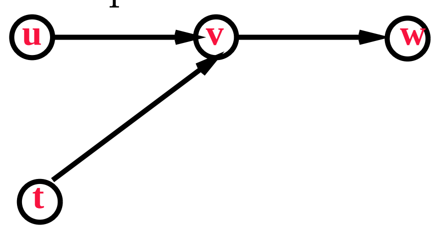




Topological Sort

Application: Scheduling

Let u represent event 1, and let v represent event 2



u must occur before v occurs

Problem: Find a schedule for executing the events that preserves the "before" relation.

Solution: Represent events as vertices and before(u,v) as a directed edge (u,v).

Let G = (V, E) be a directed, acyclic graph (DAG). If (u,v) $\in$ E, then u appears before v in the ordering of events in the schedule.

The vertices and edges in G are referred to as the topology of G.

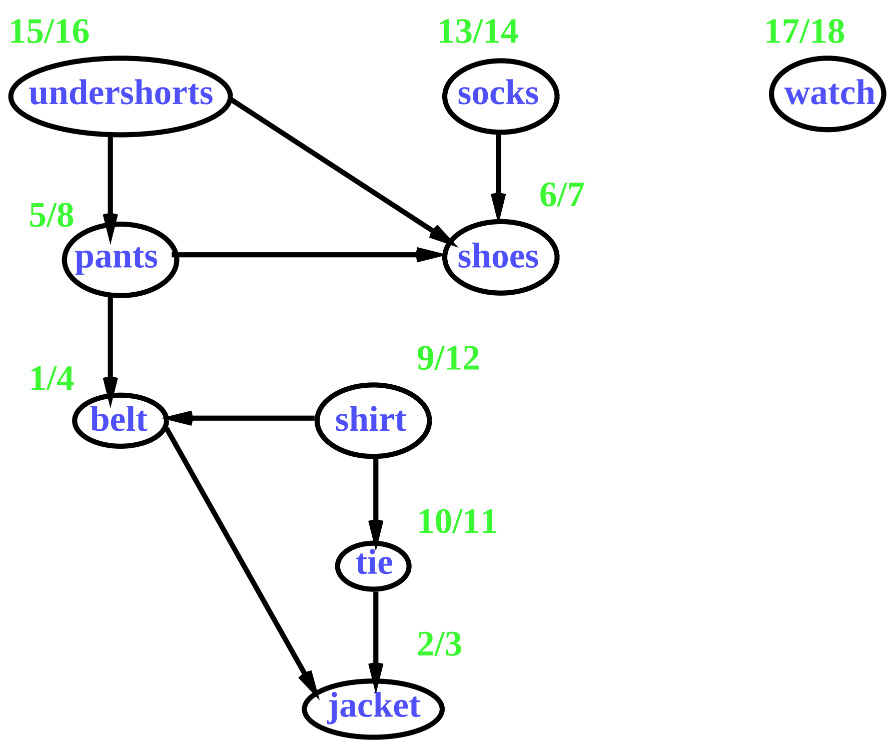
We want to sort this topology based on some key such that u $\rightarrow$ v implies key(u) $<$ key(v) (or key(v) $<$ key(u) reverse sorted).

The finish times assigned by DFS satisfy this constraint.

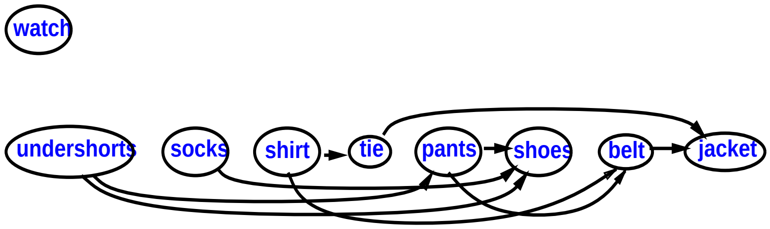
Pseudocode

```
TopologicalSort(G)              ; Θ(V + E)
   DFS(G), as each vertex finishes, insert it on the front of
          the linked list
   return linked list of vertices
```

Example: Professor Bumstead gets dressed



DFS considers vertices in alphabetical order by label.

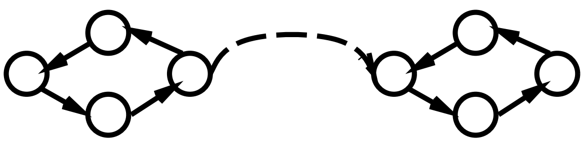


Note that a different order for DFS yields a different schedule.

Strongly Connected Components

Many graph applications look for a minimal way to connect each vertex to every other vertex.

Examples: bridging gaps, identifying bottlenecks



A graph G = (V, E) is ______________________ if for every pair of vertices <u,v>, u,v $\in$ V, there is a path ($\leadsto$) from u to v (u $\leadsto$ v) and from v to u (v $\leadsto$ u).

A __________________________________ (SCC) of a graph G = (V, E) is a maximal set U $\subseteq$ V such that for every pair <u,v> $\in$ U, u $\leadsto$ v and v $\leadsto$ u.

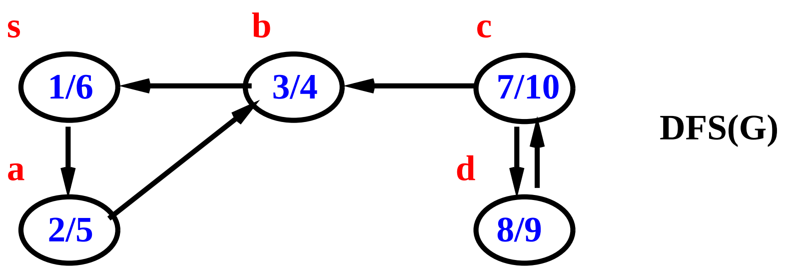
Define: The ___________ of graph G = (V, E) is the graph $G^T = (V, E^T)$, where $E^T = \{(u,v) \mid (v,u) \in E\}$.

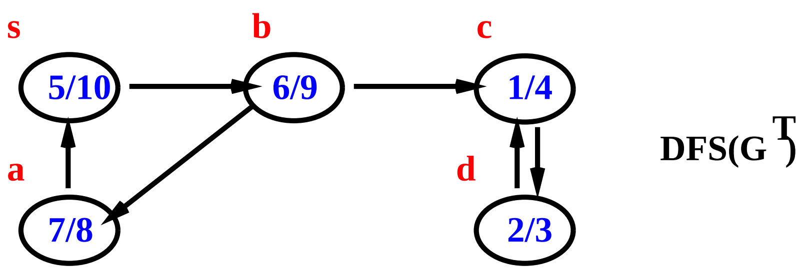
Time to create G^T = O(V+E)

Pseudocode

SCC(G)
    DFS(G) to compute finishing times
    compute G^T
    DFS(G^T) considering vertices in main loop in
        decreasing order by finish time
    output each tree in DFF of T as a SCC

Example





Applications