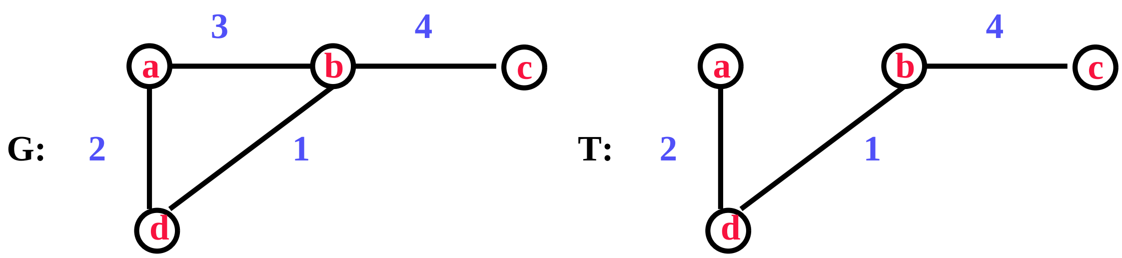
Minimum Spanning Trees

Given a connected, undirected graph G = (V, E) with edge weights w(u,v) for each edge (u,v) $\in$ E,
the ______________________________ (MST) T = (V, E') of G, $E' \subseteq E$, is an acyclic, connected graph such that w(t) = $\Sigma_{(u,v)\in E'} w(u,v)$ is minimized.

Example



Applications

Circuit wiring: connecting common pins with minimal wire

Networking

Growing a Minimal Spanning Tree

Greedy approach

Given A $\subseteq$ T = MST(G), determine a _____________ (u,v) to add to A such that A $\cup$ {(u,v)} $\subseteq$ T

Greedy-MST(G,w)

A = {}

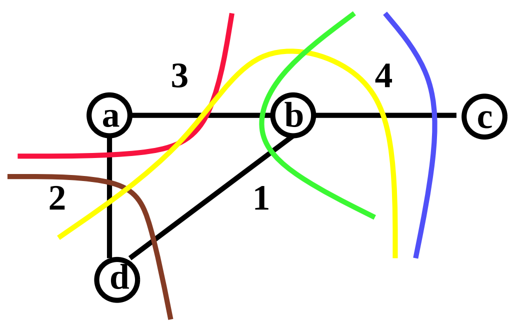
```
    while A is not a spanning tree          ; includes all vertices of G
        find a safe edge (u,v) for A
        A = A ∪ {(u,v)}
    return A
```

What is a "safe" edge?

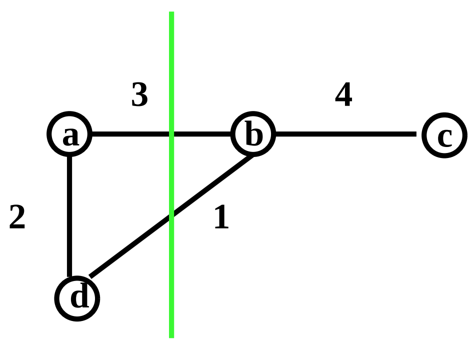
A safe edge is an edge connecting a vertex in $A \subseteq T$ to a vertex in G that is not in A such that $A \cup$ safe edge $\subseteq$ MST.

Definitions:

A ____ (S, V-S) of an undirected graph G = (V, E) is a partition of V.



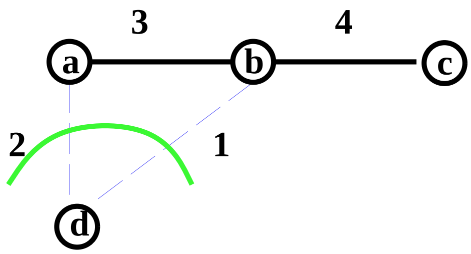
An edge $(u,v) \in E$ __________ the cut (S, V-S) if $u \in S$ and $v \in V\text{-}S$.



(a,b) and (b,d) cross the cut

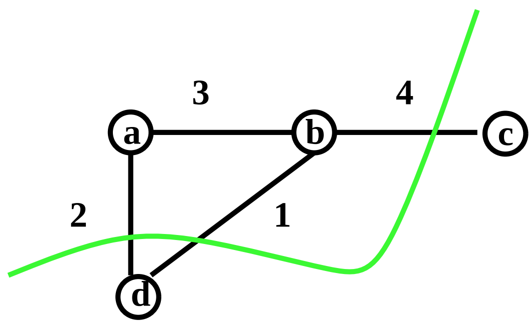
Definitions

A cut ___________ the set A of edges if no edge in A crosses the cut.



A = (V, E), V = {a, b, c, d}, E = {(a,b), (b,c)}

An edge is a ______________ crossing a cut if its weight is the minimum of any edge crossing the cut.



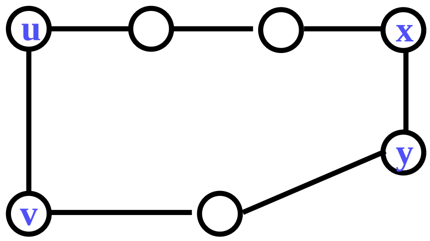
(b,d) is the light edge

Theorem 24.1

Given a connected, undirected graph G = (V, E) with edge weights w, A $\subseteq$ MST(G), cut (S, V-S) that respects A, and light edge (u,v) crossing (S, V-S), then (u,v) is a **safe edge**.

Proof: Assume T = MST(G) contains edge(x,y) crossing (S, V-S). Note that (x,y) must be on a unique path connecting u to v. Edge (u,v) would form a cycle. Removing (x,y) breaks T in 2 parts, but (u,v) reconnects them.

T' is the new resulting MST.



Since (u,v) is a light edge, then T' = T - {(x,y)} $\cup$ {(u,v)} is also MST(G).

Note that this is true because (u,v) and (x,y) cross the same cut and (u,v) is safe, w(u,v) $\leq$ w(x,y), w(T') = w(T) - w(x,y) + w(u,v) $\leq$ w(T).

Since (x,y) $\notin$ A ((S, V-S) respects A), then A $\cup$ {(u,v)} $\subseteq$ T' = MST(G). Thus, (u,v) is a safe edge.

Corollary 24.2

Given A $\subseteq$ MST(G) and a connected component C of the forest G_A(V, A), if (u,v) is a light edge connecting C to some other component in G_A, then (u,v) is safe for A.

Algorithm:

1. Find two unconnected components of G.
2. Connect them using a light edge.

Kruskal's Algorithm

Kruskal's Algorithm

    repeat

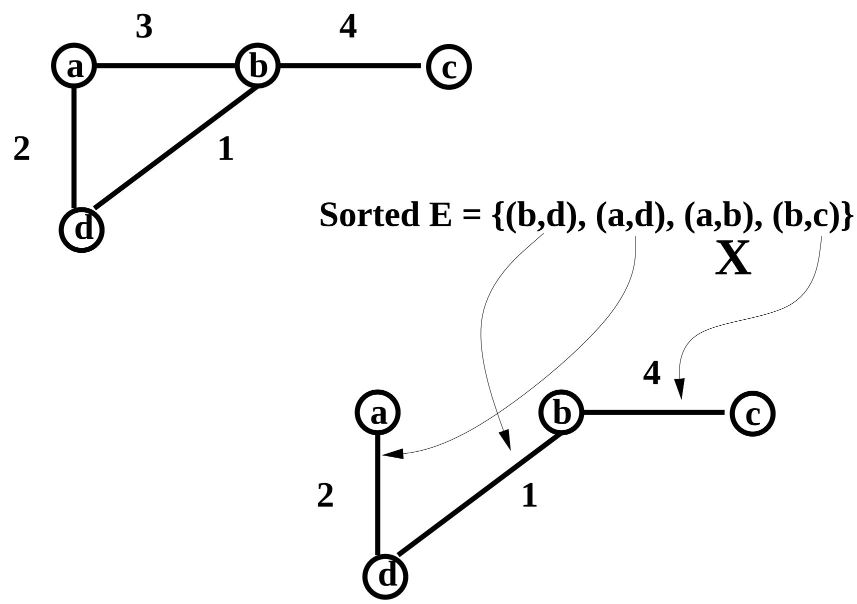
find a light edge (u,v) between two unconnected components
A = A ∪ {(u,v)}
until all edges have been considered

- Sort the edges by weight
- Use disjoint sets for speed (union by rank and path compression)

```
MST-Kruskal(G, w)                                  ; G = (V, E)
  A = {}
  foreach v in V                                   ; O(V)
      MakeSet(v)
  sort edges E by nondecreasing weight w           ; O(E lg E)
  foreach edge (u,v) in E, in order                ; m = |E| operations
      if FindSet(u) ≠ FindSet(v)                   ; n = |V| keys
      then A = A ∪ {(u,v)}                         ; O(m α(m,n))
           Union(u,v)                              ; O(E α(E,V))
  return A                                         ; α(E,V) = O(lg E)
```

$$T(V,E) = O(V) + O(E \lg E) + O(E \lg E),\ V = O(E)$$
$$= O(E \lg E)$$

Example



Prim's Algorithm

Prim's Algorithm
    repeat
        find minimal edge (u,v) connecting A to a vertex not in A
        $A = A \cup \{(u,v)\}$
    until all vertices are in A

Implementation

Maintain a priority queue Q of vertices of the form

parent	points to neighbor vertex in A along smallest edge
key	weight of smallest edge
(in Q)	true or false

Starting from some root vertex r
Update key and parent slots of vertices on A adjacent to r
Extract minimum-key vertex v from those adjacent to r
r = V

Pseudocode

```
MST-Prim(G, w, r)
foreach v in V                          ; O(V), BuildHeap
     key(v) = ∞                         ; Fibonacci Heap O(E + V lg V)
     (inQ(v) = true)
     Insert(Q, v)
key(r) = 0
parent(r) = NIL
while Q ≠ NIL                           ; O(V)
     u = Extract-Min(Q)                 ; O(lg V)
     (inQ(u) = false)                   ; Fibonacci Heap O(lg V)
     foreach v in Adj(u)                ; O(E) total
          if inQ(v) and w(u,v) < key(v) ; 2 |E|
          then parent(v) = u            ; O(lg V), DecreaseKey
               key(v) = w(u,v)          ; O(V lg V + E lg V)
```

$$O(V \lg V + E \lg V) = O(E \lg V)$$

$$\text{Fibonacci Heap: } O(E + V \lg V)$$

Example

MST-Prim(G, w, r)

Click mouse to advance to next frame.

Applications