## Maximum Flow

**Problem:** Given a network of conduits and junctions, where each conduit has a maximum capacity, calculate the maximum flow rate of material between two junctions.

**Solution:** Represent flow network as a directed graph with vertices as junctions and edges as conduit.
Edge weights are capacities.
Use ____________________ method to find maximum flow.

**Applications:** Materials processing
Assembly-line scheduling
Freight transportation

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## Flow Networks

A **flow network** G = (V, E) is a directed graph where each edge (u,v) $\in$ E has capacity c(u,v) $\geq$ 0.

If (u,v) $\notin$ E, then c(u,v) = 0.

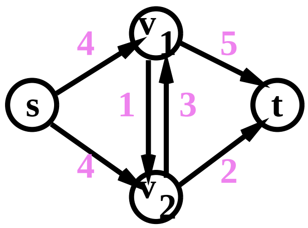
Distinguish source (s $\in$ V) and sink (t $\in$ V) as the vertices between which the flow is to be maximized.

- in-degree(s) = 0
- out-degree(t) = 0

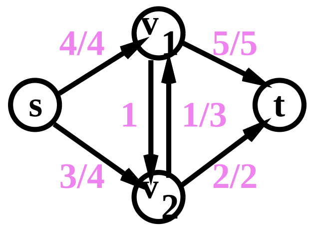
For every v $\in$ V, v is on a path from s to t. $s \leadsto v \leadsto t$, therefore the graph is connected.

$| E | \geq | V | - 1$

FLOW NETWORK




FLOW (Maximum)

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## Flow

A **flow** $f : VxV \rightarrow R$ is a real-valued function representing the rate of flow between u and v, constrained by the following:

1. ______________________________ f(u,v) $\leq$ c(u,v)
2. ____________________ f(u,v) = -f(v,u)
3. ___________________________ For every u $\in$ V - {s,t}, $\Sigma_{v \in V} f(u,v) = 0$.
   The total flow into and out of a vertex u is 0.

The value of f(u,v) is the **net flow** from vertex u to vertex v.
Note that if (u,v) and (v,u) $\notin$ E, then f(u,v) = f(v,u) = 0.
The **value** of a flow f is $| f | = \Sigma_{v \in V} f(s,v)$.

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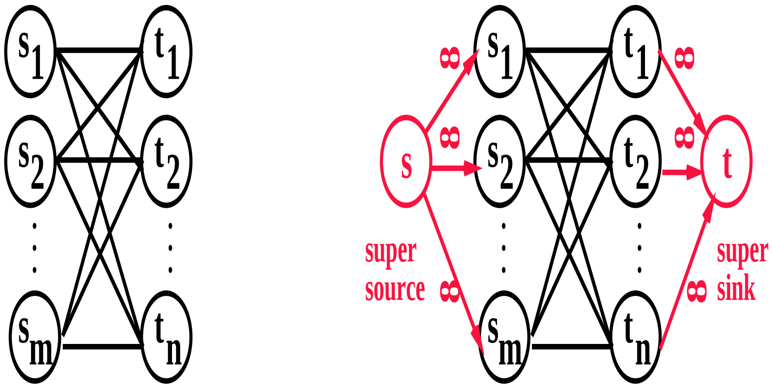
## Maximum-Flow Problem

Given flow network G, with source s and sink t, find flow with maximum value from s to t.

# Multiple Sources And Sinks




## Ford-Fulkerson Method

Look for paths from s to t that can accept more flow (augmenting path) and increase flow along this path.

```
Ford-Fulkerson-Method(G, s, t)
   flow f = 0
   while an augmenting path p exists
      augment flow f along p
   return f
```

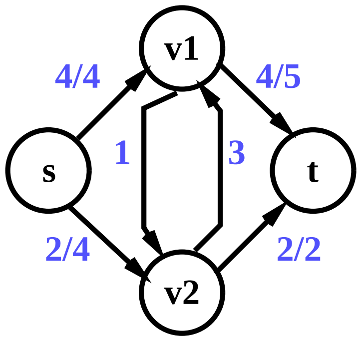
An augmenting path is a path from s to t in a residual network which consists of edges that can admit more flow.

## Residual Capacity

Given flow network G = (V, E) with flow f and two vertices u, v $\in$ V, define the **residual capacity** $c_f(u, v)$ as the additional net flow possible from u to v, not exceeding c(u,v). Thus

$$c_f(u,v) = c(u,v) - f(u,v).$$

# Example




$$c_f(s,v_2) = c(s,v_2) - f(s,v_2) = 4 - 2 = 2$$

$$c_f(v_2,t) = 2 - 2 = 0$$

$$c_f(t,v_2) = 0 - (-2) = 2$$

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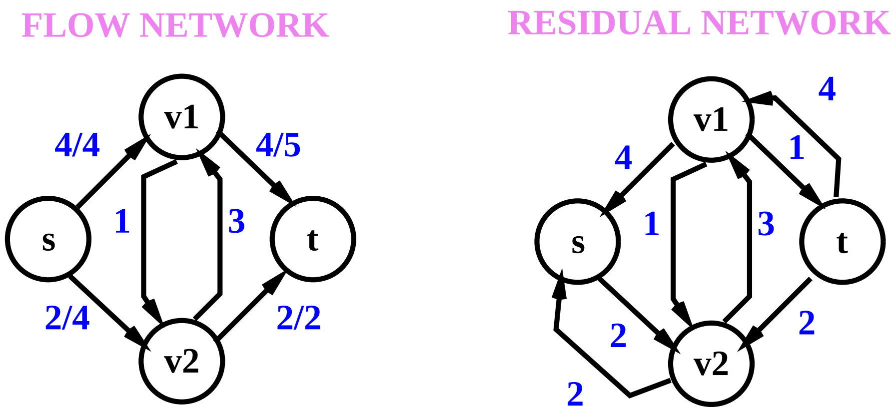
## Residual Network

Given flow network G = (V, E) and flow f, the **residual network** of G induced by f is $G_f = (V, E_f)$ where

$$E_f = \{(u,v) \in VxV \mid c_f(u,v) > 0\}$$

(u,v) $\in E_f$ is a residual edge; i.e., any edge that can admit more flow.




If there is a path from s to t in the residual network, then it is an augmenting path and indicates where flow can increase.

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## Augmenting Paths

Given flow network G and flow f, an **augmenting path** p is a simple path from s to t in the residual network $G_f$.

The minimum net flow along path p in $G_f$ indicates the amount flow can increase along this path in G.

Thus, define the **residual capacity of path p** as

$$c_f(p) \ = \ min\{c_f(u,v) \ | \ (u,v) \text{ is on } p\}$$

Define flow $f_p$ in $G_f$ as

$$f_p(u,v) \ = \ \begin{cases} c_f(p) & \text{if } (u,v) \text{ on } p \\ -c_f(p) & \text{if } (v,u) \text{ on } p \\ 0 & \text{otherwise} \end{cases}$$

$$| \ f_p \ | = \ c_f(p) \ > \ 0$$

Define **flow sum** $f_1 + f_2$ as $(f_1 + f_2)(u,v) \ = \ f_1(u,v) \ + \ f_2(u,v)$.

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### Corollary 27.4

If flow network G has flow f, augmenting path p in $G_f$, and $f' = f + f_p$, then $f'$ is a flow in G with value $| f' | = | f | + | f_p | > | f |$.

So, keep adding augmenting paths until there are no more.

## Theorem 27.2

f is a maximum flow in G if and only if residual network $G_f$ has no augmenting paths.

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### Ford-Fulkerson Algorithm

```
Ford-Fulkerson(G, s, t)          ; G = (V, E)
    foreach edge (u,v) in E
        f(u,v) = f(v,u) = 0
    while exists path p from s to t in residual network G_f
        c_f(p) = min{c_f(u,v) | (u,v) on p}
        foreach edge (u,v) on p
            f(u,v) = f(u,v) + c_f(p)
            f(v,u) = -f(u,v)
```

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### Example

Click mouse to advance to next frame.

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**Example**

Click mouse to advance to next frame.

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**Analysis:**

Depends on method for finding augmenting path.

Use breadth-first search (Edmonds-Karp Algorithm). Thus, the augmenting path is shortest from s to t.

## Theorem 27.9

The total number of augmentations is ______, ____________ per augmentation. Thus the algorithm has run time O($VE^2$).

Preflow-Push method yields O($V^2E$) (Section 27.4), and
Lift-to-Front method yields O($V^3$) (Section 27.5).

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**Applications**