Binary Search Trees

Useful for dynamic sets

Operations: Search, Min, Max, Predecessor, Successor, Insert, Delete

Performance of Some of These Operations

$\Theta(h)$, where h = height of tree

$\Theta(lgn)$, is ____________, where n is number of nodes in tree, this is true in the case of a full binary tree

$\Theta(n)$ is ______________, this is true in the case of a linear chain

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Binary Search Tree Property

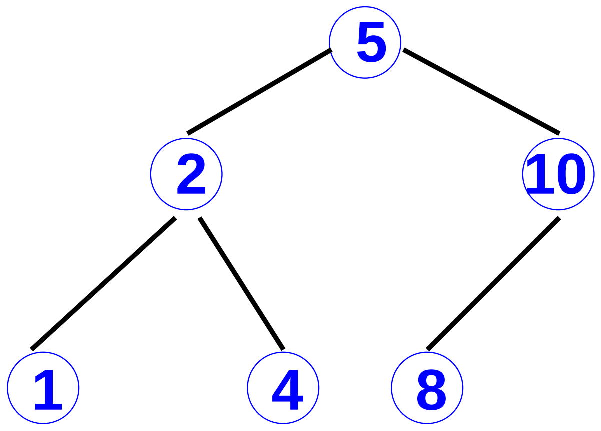
n = node in BST

l = node in left subtree of n

r = node in right subtree of n

For a binary search tree, key[l] _____ key[n] ___________ key[r]. Different from a heap, in which the left-right ordering of values does not matter.

Example



BST Traversals

Inorder(x): visit left(x), then x, then right(x)
PreOrder(x): visit x, then left(x), then right(x)
PostOrder(x): visit left(x), then right(x), then x

InOrder

```
InOrder(x)          ; prints elements in sorted order
   if x <> NIL
   then InOrder(left(x))
        print(key(x))
        InOrder(right(x))
```

InOrder traversal order: 1 2 4 5 8 10

PreOrder

```
PreOrder(x)
   if x <> NIL
   then print(key(x))
        PreOrder(left(x))
        PreOrder(right(x))
```

PreOrder traversal order: 5 2 1 4 10 8

PostOrder

```
PostOrder(x)
   if x <> NIL
   then PostOrder(left(x))
        PostOrder(right(x))
        print(key(x))
```

PostOrder traversal order: 1 4 2 8 10 5

Analysis

$$\text{T(n)} = \begin{cases} \Theta(1) & n = 0 \\ T(k) \ + \ T(n-k-1) \ + \ \Theta(1) & n > 0 \end{cases}$$

$$\begin{aligned} &\text{k = (n-1)/2:} \\ \text{T(n)} &= \text{T((n-1)/2)} + \text{T(n - (n-1)/2 - 1)} + \Theta(1) \\ &= \text{T((n-1)/2)} + \text{T(n/2 + 1/2 - 1)} + \Theta(1) \\ &= \text{T((n-1)/2)} + \text{T((n-1)/2)} + \Theta(1) \end{aligned}$$

$$= 2T((n-1)/2) + \Theta(1)$$
$$\leq 2T(n/2) + \Theta(1)$$
$$= \Theta(n)$$

$$k = 0:$$
$$T(n) = T(0) + T(n-1) + \Theta(1)$$
$$= T(n-1) + \Theta(1)$$
$$= \Theta(n)$$

Searching

```
Search(n, k)                ; n is a pointer to a node, not the tr
   if n=NIL or k=key(n)     ; Initially n points to the root node
   then return n
   if k < key(n)
   then return Search(left(n), k)
   else return Search(right(n), k)
```

Example: Search(Root(T), 14)

Analysis

Could write this code iteratively:

```
   if k < key(n)
   then n = left(n)
   else n = right(n)
   LOOP
```

- This code is clearly $\Theta(h)$.
- Not necessarily $\Theta(lgn)$.
- Remember, we are not assured that the tree is balanced

```
Min(n)     ; leftmost leaf of tree rooted at n
   while left(n) <> NIL
      n = left(n)
   return n
```

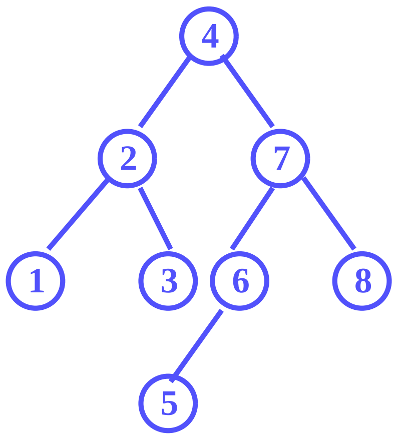
Min is ______

```
Max(n)     ; rightmost leaf of tree rooted at n
   while right(n) <> NIL
      n = right(n)
   return n
```

Max is ______

Successor(n)

Case I:
Try n=4
right(n) $\neq$ NIL
return(Min(right(n)))



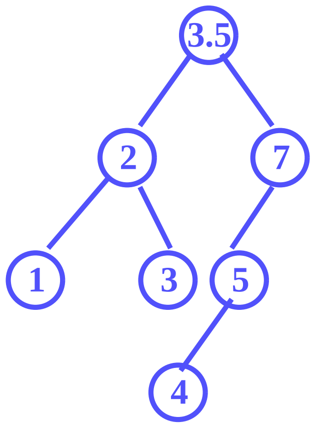
Try n=7

Try n=5

Successor(n)

right(n) = NIL

Case II: Find s such that left(s) is n or an ancestor of n



Try n=5

Case III: No such ancestor exists; thus, no successor

Try n=7

Successor(n)

```
Successor(n)
   if right(n) <> NIL
   then return Min(right(n))
   p = Parent(n)
   while p <> NIL and n = right(p)
      n = p
      p = Parent(p)
   return(p)
```

Successor is ______

```
Predecessor(n)
   if left(n) <> NIL
   then return Max(left(n))
   p = Parent(n)
   while p <> NIL and n = left(p)
      n = p
      p = Parent(p)
   return(p)
```

Predecessor is ______

Insertion

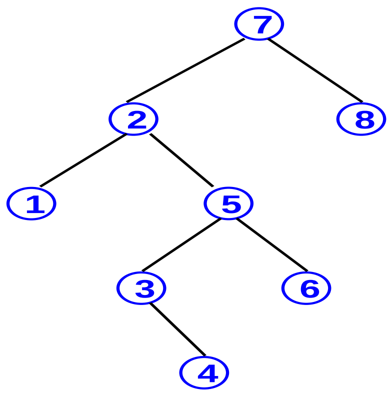
1. Go search for key until run off the end of the tree
2. Put new key there

```
Insert(T,x)                             ; x = pointer to new node
   p = NIL
   n = root(T)                          ; Search for key
   while n <> NIL
      p = n
      if key(x) < key(n)
      then n = left(n)
      else n = right(n)
   Parent(x) = p                        ; Insert new key
   if p = NIL
   then root(T) = x
   else if key(x) < key(p)
      then left(p) = x
      else right(p) = x
```

Insert is ______

Examples

Insert(T, 6)



For a sequence of integers in the range $1 \ldots 10$, which insertion order will yield the tallest tree? ______________________________

Which insertion order will yield the shortest tree? ______________________

Delete

Case I: Node is a leaf. ____________________

Delete

Case II: Node has only one child. _____________________

Delete

Case III: Node has two children. Delete ____________ of node and replace node's key with successor's key. Note that successor always satisfies ________________________

Pseudocode

```
Delete(T,x)                                    ; Returns deleted node
  if left(x) = NIL or right(x) = NIL
  then d = x                                   ; Case I and II
  else d = Successor(x)                        ; Case III
  if left(d) <> NIL                            ; Get child of node to b
  then c = left(d)
```

```
else c = right(d)
if c <> NIL
then Parent(c) = Parent(d)                 ; Remove node d
if Parent(d) = NIL
then root(T) = c
else if d = left(parent(d))
     then left(parent(d)) = c
     else right(parent(d)) = c
if d <> x                                  ; Case III
then key(x) = key(d)
return(d)
```

Randomly Built BSTs

Theorem 13.6: The average height of a randomly built BST on n distinct keys is __________ By randomly built we mean that keys are inserted in random order.

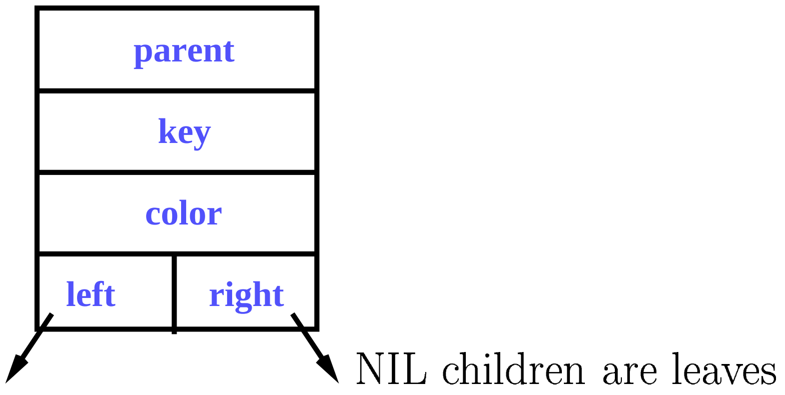
Red-Black Trees

Binary Search Tree operations are ______ not _______________

If we can keep the tree balanced, the operations will be _________

This is the goal of Red-Black trees.

- Internal Node



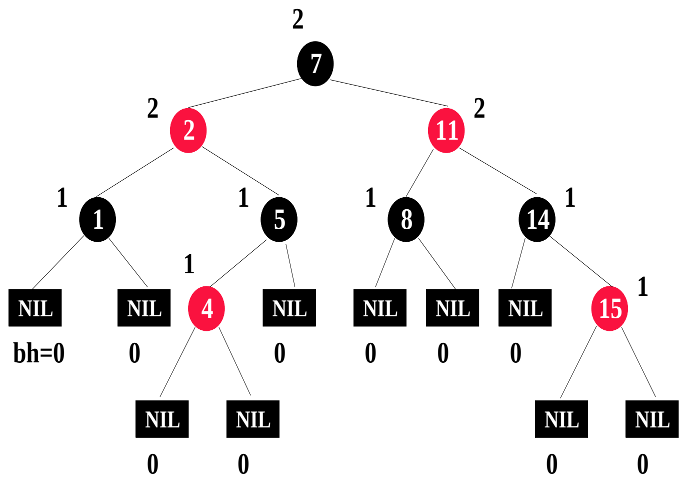
  color $\in$ {Red, Black}

- Leaf Node: NIL
  Leaf nodes are sentinels (dummy objects), and the color is always Black.

By constraining colors of nodes along paths from root to leaf, RB trees ensure no path is more than twice as long as any other (the tree is balanced).

Properties of RB Trees

1. Every node is either _____ or ____________.
2. Every leaf is ________.
3. If a node is Red, then both its children are ________.
4. Every path from some node to a leaf contains the _________________ of Black nodes.



Properties of RB Trees

Definition: The ________________ of a node n, denoted bh(n), is the number of black nodes (excluding n) on the path from n to a leaf, including the leaf.

bh(root) = Black Height of the tree

By property 4, bh(n) is the same regardless of the path.

Lemma 14.1

A RB tree with n internal nodes has height at most ____________.

Thus the dynamic set operations on RB trees are all ________.

Proof:

1. First show that subtree rooted at x contains at least $2^{bh(x)} - 1$ internal nodes.

Proof by induction.

Initial condition: if height(x) = 0, then x is a leaf whose subtree contains at least $2^{bh(x)} - 1 = 2^0$ - 1 = 0 internal nodes.

Inductive Step: Consider internal node x. Each child has black-height bh(x) (if the child is Red) or bh(x)-1 (if the child is Black).

By the Inductive Hypothesis, the child has at least $2^{bh(x)-1} - 1$ internal nodes.

Therefore the subtree rooted at x has at least $(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1$ internal nodes, or $2^{bh(x)} - 1$ internal nodes.

2. Next, by property 3, a tree of height h has a black-height of at least h/2.

$$\begin{gathered} bh(T) \geq h/2 \\ n \geq 2^{bh(T)} - 1 \\ n \geq 2^{h/2} - 1,\ n+1 \geq 2^{h/2} \\ \lg(n+1) \geq \lg(2^{h/2}) \\ \lg(n+1) \geq h/2 \\ h \leq 2\lg(n+1) \end{gathered}$$

Properties of RB Trees

As you can tell, Insert and Delete need some work to maintain the RB tree properties.

Question: Is a RB tree with a Red root still a RB tree if we change the root color to Black?

Answer: ____

Rotations

Redistribute nodes in the tree.

Rotations

```
Left-Rotate(T,x)
   y = right(x)                        ; assume right(x) <> NIL
   right(x) = left(y)                  ; move y's child over
   if left(y) <> NIL
   then parent(left(y)) = x
   parent(y) = parent(x)               ; move y up to x's positi
   if parent(x) = NIL
   then root(T) = y
   else if x = left(parent(x))
        then left(parent(x)) = y
        else right(parent(x)) = y
   left(y) = x                         ; move x down
   parent(x) = y
```

Rotations

```
Right-Rotate(T,y)
   x = left(y)               ; assume left(y) <> NIL
   left(y) = right(x)
   if right(x) <> NIL
   then parent(right(x)) = y
```

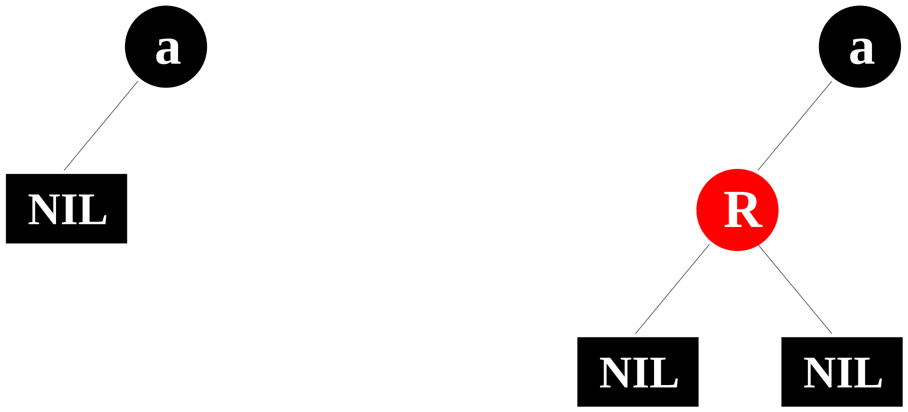
```
parent(x) = parent(y)
if parent(y) = NIL
then root(T) = x
else if y = left(parent(y))
     then left(parent(y)) = x
     else right(parent(y)) = x
right(x) = y
parent(y) = x
```

Insertion

1. Insert node into tree using BST Insert(T,x) and color node Red
2. Fix violated RBT properties
3. Color root Black

Which properties might be violated?

1. _____, new node is Red; previous nodes are already colored
2. _____, new node inserted with NIL (Black) leaves
3. __________, parent may also be Red
4. _____, replacing Black node with a Red and Black node

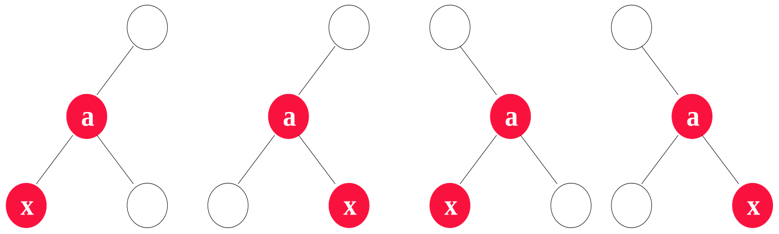


- Tree was balanced before insert
- If colored Black, may violate property 4

RB Trees

If parent node ‘a’ was Black, then no changes are necessary.

If not, then there are three cases to consider for each of the orientations below.



3 cases to consider for each of these

Move up the tree until there are no violations or we are at the root.

In the following discussion we will assume the parent is a left child (if the parent is a right child perform the same steps swapping “right” and “left”)

RB-Insert(T,x)

Case I: x's uncle is Red

```
if parent(x) = left(parent(parent(x)))
then uncle(x) = right(parent(parent(x)))
else uncle(x) = left(parent(parent(x)))
```

- Change x's grandparent to Red
- Change x's uncle and parent to Black
- Change x to x's grandparent

Case II: x's uncle is Black, x is the right child of its parent

- Change x to x's parent
- Rotate x's parent (now x) left to make Case III
- Case II is now Case III

Click mouse to advance to next frame.

Case III: x's uncle is Black, x is the left child of its parent

- Set x's parent to Black
- Set x's grandparent to Red
- Rotate x's grandparent right

Click mouse to advance to next frame.

Pseudocode

```
RB-Insert(T,x)
   Insert(T,x)
   color(x) = Red
   while x <> root(T) and color(parent(x)) = Red
      if parent(x) = left(parent(parent(x)))
      then uncle = right(parent(parent(x)))
         if color(uncle) = Red
         then color(parent(x)) = Black                    ; Ca
              color(uncle) = Black
              color(parent(parent(x))) = Red
              x = parent(parent(x))
         else if x = right(parent(x))
              then x = parent(x)                          ; Ca
                   Left-Rotate(T,x)
              color(parent(x)) = Black                    ; Ca
              color(parent(parent(x)) = Red
              Right-Rotate(T, parent(parent(x)))
      else
         ...  ; same as then with "right" and "left" swapped
   color(root(x)) = Black
```

The performance of this algorithm is O(lgn), with ≤ 2 rotations

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Deletion

1. Delete node from tree using an algorithm similar to BST Delete(T,x)
2. Fix violated properties

Which properties might be violated?

- If node deleted was Red, _____
- If node deleted was Black, then property _ will be violated
- Property _ is also violated but is immediately fixed

To correct the violations, look at the violation from another perspective: Assume the child of the deleted node is colored “double-Black”, violating property 1, and we want to give half of the “double-Black” to another Red node or push half of the Black out the top of the tree.

Sortof BST Delete

1. Use of sentinel nil(T) for NIL leaves
2. Call to RB-Delete-Fixup

```
RB-Delete(R,z)                  ; return deleted node
   if left(z) = nil(T) or right(z) = nil(T)
   then d = z
   else d = Successor(z)
   if left(d) <> nil(T)
   then c = left(d)
   else c = right(d)
   parent(c) = parent(d)        ; no test for NIL with sentine
   if parent(d) = nil(T)
   then root(T) = c
   else if d = left(parent(d))
        then left(parent(d)) = c
        else right(parent(d)) = c
```

```
if d <> z
then key(z) = key(d)
if color(d) = Black
then RB-Delete-Fixup(T,c)  ; c is now "Double-Black"
return d
```

RB-Delete-Fixup(T,x)

First, if color(x) = Red, then color x Black; done!

Note that x always has a sibling s. This is because if x is Black and is the child of a deleted Black node, then there is a sibling s because the tree was previously balanced.

There are four cases to consider for each orientation of x (whether x is a left child or a right child). In each case we need to maintain the number of Black nodes.

For these examples we will assume that x = left(parent(x)) (x is a left child).

Case I:

x's sibling is Red, s has two Black children

- Switch colors of s and parent(x) (color(s) = Black, color(parent(x)) = Red)
- Rotate parent(x) left
- Reset sibling s

- Case I is now Case II, Case III, or Case IV

Case II:

Sibling is black, sibling's children are both Black

- Change s to Red
- Add extra Black to parent(x)
- Repeat while loop with parent(x)
- If entered Case II from Case I, will then terminate (parent(x) = Red)

Case III:

x's sibling is Black, s's left child is Red, s's right child is Black

- Switch colors of s (Red) and left(s) (Black)
- Rotate s right
- Reset sibling s
- Case III is now Case IV

Case IV:

x's sibling is black, s's right child is Red

- Change color(s) to color of parent(x)

- Change color of parent to Black
- Change color of sibling's right child (Red) to Black
- Rotate parent(x) left
- All done!

Pseudocode

```
RB-Delete-Fixup(T,x)
   while x <> root(T) and color(x) = Black
      if x = left(parent(x))
      then s = right(parent(x))                ; Get x's sibl
           if color(s) = Red
           then color(s) = Black               ; Case I
                color(parent(x)) = Red
                Left-Rotate(T, parent(x))
                s = right(parent(x))
           if color(left(s)) = Black and color(right(s)) = B
           then color(s) = Red                 ; Case II
                x = parent(x)
           else if color(right(s)) = Black
                then color(left(s)) = Black    ; Case III
                     color(s) = Red
                     Right-Rotate(T,s)
                     s = right(parent(x))
                color(s) = color(parent(x))    ; Case IV
                color(parent(x)) = Black
                color(right(s)) = Black
```

```
                    Left-Rotate(T, parent(x))
                    x = root(T)
          else
             ...          ; Same as then with right and left swap
    color(x) = Black
```

The performance of this algorithm is O(lgn), with ≤ 3 rotations
Thus we see that RB trees maintain O(lgn) time for dynamic-set operations.

Applications