Evaluation of Multidimensional Gaussian PDF Over Half-Plane

EE 451 Spring 2009

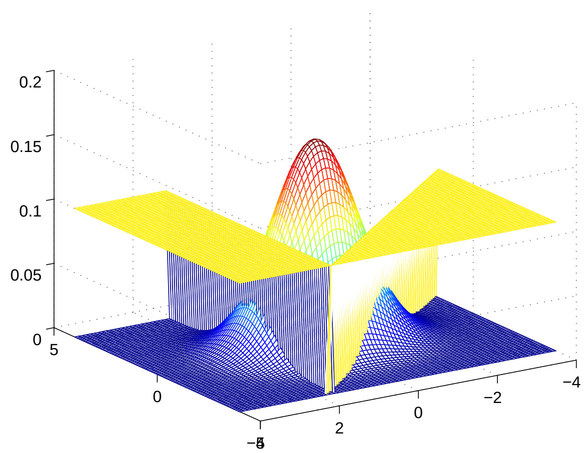
Let $\mathbf{x}$ be a n-dimensional zero-mean Gaussian vector with covariance $\Lambda = \sigma^2 I$. That is, the dimensions of $\mathbf{x}$ are independent and identically distributed (iid) zero-mean Gaussian random variables with variance σ^2. The probability density function (pdf) of each dimension is

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

and the joint pdf for the vector is

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} p_X(x_i) = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}}.$$

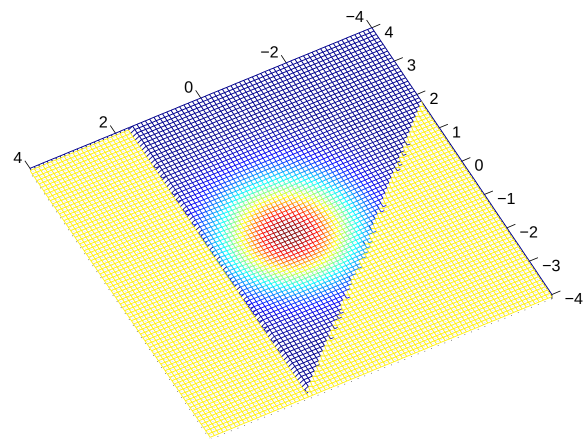
Now, consider the problem of integrating the Gaussian vector pdf over a half-space at distance d from the the mean of $\mathbf{x}$ (that is, a distance d from the origin). The situation is described in the figure below for the case $n = 2$ and two half-planes at distance $d = 1.6$ from the origin.



One half-plane is defined by a line at $x_1 = d$ and parallel to the vertical axis. Let S denote this half-plane. The other half-plane is at a 45 degree angle to the x_1, x_2 axes. The integral over the two half-planes yields the same value. Integrating over the first half-plane yields

$$\int\int_{\mathbf{x}\in S} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int_{d}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_1^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_2^2}{2\sigma^2}} dx_1 dx_2 = Q(\frac{d}{\sigma}).$$

The figure below shows an overhead view of the figure above. The 2-dimensional Gaussian pdf is circularly symmetric, the two lines defining the half-planes are at a distance d from the origin (from the mean of the random vector) and it is evident from the two figures that the integration of the Gaussian pdf over the half-planes must yield the same value.



The Matlab function halfplane.m available at
www.eecs.wsu.edu/∼fischer/ee451material.html
allows simple generation of the figures above. The mesh plot can then be rotated to better examine the intersection of the half-planes and the Gaussian pdf.

The general conclusion is that if S is a half-space at distance d from the mean of an iid Gaussian random vector, then

$$\int \cdots \int_{\mathbf{x} \in S} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = Q(\frac{d}{\sigma}).$$

The Q-function can be evaluated numerically or approximated. Matlab contains two functions that can be used for numerical evaluation: qfunc(x) directly evaluates $Q(x)$. Also, the complementary error function can be used, as $Q(x)$ is also calculated as $0.5\text{erfc}(x/\sqrt{2})$. The figure below plots the Q-function together with useful upper and lower bounds given by

a) Upper bound

$$Q_U = \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$$

b) Lower bound

$$Q_L = \frac{1}{\sqrt{2\pi}x}(1 - \frac{1}{x^2})e^{-x^2/2}.$$