Decentralized Control for Output Synchronization in Heterogeneous Networks of Non-Introspective Agents

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***Abstract*— In this paper we consider the output synchronization problem for heterogeneous networks of linear agents. The network provides each agent with a linear combination of its own output relative to that of neighboring agents, and it allows the agents to exchange information about their own internal observer estimates. We design decentralized controllers based on setting the control input of a single root agent to zero and letting the remaining agents synchronize to the root agent. A distinguishing feature of our work is that the agents are assumed to be *non-introspective*, meaning that they possess no knowledge about their own state or output separate from what is received via the network.**

I. Introduction

The problem of achieving *synchronization* among agents in a network—that is, asymptotic agreement on the agents' state or output trajectories—has received substantial attention in recent years. The essential difficulty of the synchronization problem is the lack of a central authority with the ability to control the network as a whole. Instead, each agent must implement a controller based on limited information about itself and its surroundings—typically in the form of measurements of its own state or output relative to that of neighboring agents in the network.

Much of the attention has been directed toward *state synchronization* in *homogeneous* networks (that is, networks where the agents are described by identical models) where each agent receives information about its own state relative to that of neighboring agents [1]–[5]. Roy, Saberi, and Herlugson [6], Tuna [7], and Yang, Roy, Wan, and Saberi [8] considered more general observation topologies and more complex identical agents than previously considered. Others have studied the case where the agents receive relative information about their own partial-state output [9]–[13]. In this context, Li, Duan, Chen, and Huang [12] introduced the idea of a distributed observer, which makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates.

Many of the results on the synchronization problem are rooted in the seminal work of Wu and Chua [14], [15].

A. Heterogeneous Networks and Output Synchronization

A limited amount of research has also been conducted on *heterogeneous* networks (that is, networks where the agent models are non-identical). Ramírez and Femat [16] presented a robust state-synchronization design for networks of nonlinear systems with relative degree one, where each agent implements a sufficiently strong feedback based on the difference between its own state and that of a common reference model. In the work of Xiang and Chen [17] it is assumed that a common Lyapunov function candidate is available, which is used to analyze stability with respect to a common equilibrium point. Depending on the system, some agents may also implement feedbacks to ensure stability, based on the difference between those agents' states and the equilibrium point. Zhao, Hill, and Liu [18] analyzed state synchronization in a network of nonlinear agents based on the network topology and the existence of certain time-varying matrices. Controllers can be designed based on this analysis, to the extent that the available information and actuation allows for the necessary manipulation of the network topology.

In heterogeneous networks, the physical interpretation of one agent's internal state may be different from that of another agent. Indeed, the agents may be governed by models of different dimensions. In this case, comparing the agents' internal states is not meaningful, and it is more natural to aim for *output synchronization*—that is, agreement on some partial-state output from each agent. Chopra and Spong [19] focused on output synchronization for weakly minimum-phase systems of relative degree one, using a pre-feedback within each agent to create a single-integrator system with decoupled zero dynamics. Pre-feedbacks were also used by Bai, Arcak, and Wen [20] to facilitate passivity-based designs. The authors have considered output synchronization for right-invertible agents by using pre-compensators and an observer-based pre-feedback within each agent to yield a network of asymptotically identical agents [21].

Kim, Shim, and Seo [22] studied output synchronization for uncertain single-input single-output, minimum-phase systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. A similar approach was taken by Wieland, Sepulchre, and Allgöwer [23], who showed that a necessary condition for output synchronization in heterogeneous networks is the existence of a *virtual exosystem* that produces a trajectory to which all the agents asymptotically converge. If one knows the model of an observable virtual exosystem without exponentially unstable modes, which each agent is capable of tracking, then it can be implemented within each agent

The work of Håvard Fjær Grip is supported by the Research Council of Norway. The work of Tao Yang and Ali Saberi is partially supported by National Science Foundation grant NSF-0901137 and NAVY grants ONR KKK777SB001 and ONR KKK760SB0012.

and synchronized via the network. The agent can then be made to track the model with the help of a local observer estimating the agent's states.

B. Introspective Versus Non-Introspective Agents

The designs mentioned in the previous section rely—explicitly or implicitly—on some sort of self-knowledge that is separate from the information transmitted over the network. In particular, the agents may be required to know their own state, their own output, or their own state/output relative to that of a reference trajectory. In this paper we shall refer to agents that possess this type of self-knowledge as *introspective* agents, to distinguish them from *non-introspective* agents—that is, agents that have no knowledge of their own state or output separate from what is received via the network. This distinction is significant because introspective agents have much greater freedom to manipulate their internal dynamics and thus change the way that they present themselves to the rest of the network (e.g., through the use of pre-compensators). The notion of a non-introspective agent is also practically relevant; for example, two vehicles in close proximity may be able to measure their relative distance without either of them having knowledge of their absolute position.

To the authors' knowledge, the only result that solves the output synchronization problem for a well-defined class of heterogeneous networks with non-introspective agents is by Zhao, Hill, and Liu [24]. In their work, the only information available to each agent is a linear combination of outputs received over the network. However, the agents are assumed to be passive—a strict requirement that, among other things, requires the agents to be weakly minimum-phase and of relative degree one.

C. Contributions of This Paper

In this paper we consider heterogeneous networks of non-introspective linear agents that receive, via the network's communication infrastructure, a linear combination of their own output relative to that of neighboring agents. In the spirit of Li et al. [12] we also assume that the agents can exchange information about their internal estimates using the same infrastructure. We design decentralized controllers that achieve output synchronization under a set of straightforward assumptions about the agents and the network topology.

D. Notation

For a matrix A, A' denotes its transpose and A^* denotes its conjugate transpose. The Kronecker product between A and B is denoted by $A \otimes B$.

II. Problem Formulation

We consider a network of n multiple-input multiple-output agents on the form

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{1a}$$

$$y_i = C_i x_i + D_i u_i, \tag{1b}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$. Our goal is to achieve output synchronization among the agents, meaning that $\lim_{t\to\infty}(y_i - y_j) = 0$ for all $i, j \in \{1, \dots, n\}$.

The agents are non-introspective; hence, agent i does not have access to its own output y_i. The only available information comes from the network, which provides each agent with a linear combination of its own output relative to that of the other agents. In particular, agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{n} a_{ij}(y_i - y_j),$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$. The communication topology of the network can be described by a directed graph (digraph) $\mathscr{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i. Agent j is then called a *parent* of agent i, and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of a_{ij}.

We shall frequently make use of the matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{n} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. This matrix is known as the *Laplacian* matrix of $\mathscr{G}$ and has the property that all the row sums are zero. In terms of the coefficients of G, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{n} g_{ij} y_j.$$

We also assume that the agents can exchange information about their internal estimates using the same communication infrastructure. Specifically, agent i is presumed to have access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{n} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{n} g_{ij}\eta_j,$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced internally by agent j. This variable will be specified as we proceed with the design.

A. Assumptions

We make the following assumptions about the network topology and the individual agents.

Assumption 1: The digraph $\mathscr{G}$ has a directed spanning tree with root agent $K \in \{1, \dots, n\}$, such that for each $i \in \{1, \dots, n\} \setminus K$,

1) (A_i, B_i) is stabilizable
2) (A_i, C_i) is observable
3) (A_i, B_i, C_i, D_i) is right-invertible
4) (A_i, B_i, C_i, D_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of A_K

Remark 1: A *directed spanning tree* is a directed subgraph of $\mathscr{G}$, consisting of all the nodes and a subset of the edges, such that every node has exactly one parent, except a single root node with no parents. Furthermore, there must exist a directed path from the root to every other agent. A digraph

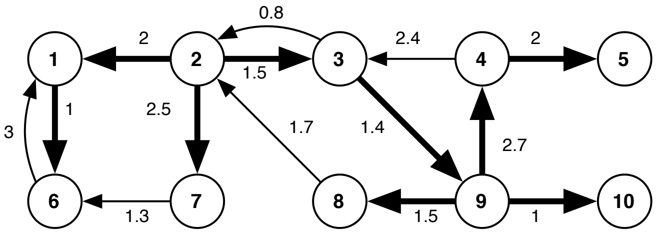

Fig. 1. The depicted digraph contains multiple directed spanning trees, rooted at nodes 2, 3, 4, 8, and 9. One of these, with root node 2, is illustrated by bold arrows.

may contain many directed spanning trees, and thus there may be several choices of root agent K. Fig. 1 illustrates an example digraph containing multiple directed spanning trees.

Remark 2: Right-invertibility of a system on the form (1) means that, given a reference output $y_r(t)$ for $t \geq 0$, there exist an initial condition $x_i(0)$ and an input $u_i(t)$ such that $y_i(t) = y_r(t)$ for all $t \geq 0$. For example, every single-input single-output system is right-invertible, unless its transfer function is identically zero.

We shall need the following result, which is proven in Appendix I.

Lemma 1: Let $\bar{G}$ be defined by removing the K'th column and row from G. Then all the eigenvalues of $\bar{G}$ are in the open right-half complex plane.

III. Control Design

In this section we describe the construction of decentralized controllers that achieve output synchronization. We shall first describe the motivation behind the design.

The main idea is to set the control input of the root agent K to zero (i.e., $u_K = 0$), and to also select $\eta_K = 0$. We then design controllers for all the other agents such that their outputs asymptotically synchronize with the trajectory $y_K(t)$. That is, for each $i \in \{1, \ldots, n\} \setminus K$ we wish to regulate the synchronization error variable

$$e_i := y_i - y_K$$

to zero, where the dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x_i \\ x_K \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{2a}$$

$$e_i = \begin{bmatrix} C_i & -C_K \end{bmatrix} \begin{bmatrix} x_i \\ x_K \end{bmatrix} + D_i u_i. \tag{2b}$$

To achieve our objective, we carry out the design for each agent $i \in \{1, \ldots, n\} \setminus K$ in three steps.

In Step 1 we construct a new state $\bar{x}_i$, via a transformation of x_i and x_K, so that the dynamics of the synchronization error variable e_i can be described by the alternative equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{3a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i u_i := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i u_i. \tag{3b}$$

The purpose of this state transformation is to reduce the dimension of the model underlying e_i by removing redundant modes that have no effect on e_i. In particular, the model (2) may be unobservable, but the model (3) is always observable.

The properties of the model (3) also allow us, in Step 2 of the design, to construct a controller that regulates e_i to zero by using state feedback from $\bar{x}_i$. This controller is not directly implementable, however, because $\bar{x}_i$ is not known to agent i. This brings us to Step 3 of the design.

In Step 3 we construct a decentralized observer that makes an estimate of $\bar{x}_i$ available to agent i. The design of this observer is based on previous results on distributed observer design for *homogeneous* networks. Since our network is heterogeneous, we first perform a second state transformation, from $\bar{x}_i$ to χ_i, in order to obtain a set of dynamical models that are substantially the same for all the agents. In particular, the model differences now occur only in particular locations where they can be suppressed by using high-gain observer techniques. By combining the observer estimates with the state-feedback controllers designed in Step 2, we achieve output synchronization.

A. Preliminaries

Because we choose $u_K = 0$, the trajectory $y_K(t)$ is the unforced response of agent K, consisting of a linear combination of the observable modes of the pair (A_K, C_K). Asymptotically stable modes vanish as $t \to \infty$, and they therefore play no role asymptotically. For simplicity of presentation, we therefore assume without loss of generality that all the eigenvalues of A_K are in the closed right-half plane and that (A_K, C_K) is observable.

Below we describe the three steps of the design procedure that must be carried out for each agent $i \in \{1, \ldots, n\} \setminus K$. In addition to agent i's system matrices (A_i, B_i, C_i, D_i), the information needed to carry out these three steps is as follows:

- the matrices A_K and C_K of the root node
- an integer $\bar{n}$ such that $\bar{n} \geq n_i + n_K$ for all $i \in \{1, \ldots, n\} \setminus K$.
- a number $\tau > 0$ that is a lower bound on the real part of the eigenvalues of the matrix $\bar{G}$ defined in Lemma 1
- a common high-gain parameter $\varepsilon \in (0, 1]$

B. Design Procedure for Agents $i \in \{1, \ldots, n\} \setminus K$

Step 1: State Transformation: Let O_i be the observability matrix corresponding to the system (2):

$$O_i = \begin{bmatrix} C_i & -C_K \\ \vdots & \vdots \\ C_i A_i^{n_i+n_K-1} & -C_K A_K^{n_i+n_K-1} \end{bmatrix}. \tag{4}$$

Let q_i denote the dimension of the null space of O_i, and define $r_i = n_K - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{n_K \times q_i}$ such that

$$O_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \quad \operatorname{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i. \tag{5}$$

Because (A_i, C_i) and (A_K, C_K) are observable, it is easy to show that Λ_{iu} and Φ_{iu} have full column rank. Let therefore Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{n_K \times n_K}$ are nonsingular.

We define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i + r_i}$ as

$$\bar{x}_i = \begin{bmatrix} x_i - \Lambda_i M_i \Phi_i^{-1} x_K \\ -N_i \Phi_i^{-1} x_K \end{bmatrix}.$$

where $M_i \in \mathbb{R}^{n_i \times n_K}$ and $N_i \in \mathbb{R}^{r_i \times n_K}$ are defined as

$$M_i = \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 & I_{r_i} \end{bmatrix}.$$

The following lemma, which is proven in Appendix I, shows how the synchronization error e_i is given in terms of $\bar{x}_i$.

Lemma 2: The synchronization error variable e_i is governed by dynamical equations of the form (3), where $(\bar{A}_i, \bar{C}_i)$ is observable and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K.

Step 2: State-Feedback Control Design: We now design a controller as a function of $\bar{x}_i$ to regulate e_i to zero. Consider the following equations with unknowns $\Pi_i \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i \in \mathbb{R}^{m_i \times r_i}$, commonly known as the *regulator equations*:

$$\Pi_i \bar{A}_{i22} = A_i \Pi_i + \bar{A}_{i12} + B_i \Gamma_i, \tag{6a}$$

$$C_i \Pi_i - \bar{C}_{i2} + D_i \Gamma_i = 0. \tag{6b}$$

Based on Π_i and Γ_i, we define a matrix

$$\bar{F}_i = \begin{bmatrix} F_i & \Gamma_i - F_i \Pi_i \end{bmatrix}, \tag{7}$$

where F_i is chosen such that $A_i + B_i F_i$ is Hurwitz. The following lemma, which is proven in Appendix I, shows that the regulator equations (6) are always solvable and that the matrix $\bar{F}_i$ can be used to define a state-feedback controller.

Lemma 3: The regulator equations (6) are solvable, and the state-feedback controller $u_i = \bar{F}_i \bar{x}_i$ ensures that $\lim_{t\to\infty} e_i = 0$.

Step 3: Observer-Based Implementation: Our last step is to design an observer to produce an estimate of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$. To this end, define $\chi_i = T_i \bar{x}_i$, where

$$T_i = \begin{bmatrix} \bar{C}_i \\ \vdots \\ \bar{C}_i \bar{A}_i^{\bar{n}-1} \end{bmatrix}.$$

Note that T_i is generally not a square matrix; however, due to observability of $(\bar{A}_i, \bar{C}_i)$, T_i is injective, which implies that $T_i' T_i$ is nonsingular. In terms of the new state χ_i, we can write the system equations as

$$\dot{\chi}_i = (\mathcal{A} + \mathcal{L}_i)\chi_i + \mathcal{B}_i u_i, \tag{8a}$$

$$e_i = \mathcal{C}\chi_i + \mathcal{D}_i u_i, \tag{8b}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

$$\mathcal{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathcal{B}_i = T_i \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \mathcal{D}_i = D_i,$$

for some matrix $L_i \in \mathbb{R}^{p \times \bar{n}p}$. Note that the matrices $\mathcal{A}$ and $\mathcal{C}$ are the same for all the agents, and the special form of these matrices imply that $(\mathcal{A}, \mathcal{C})$ is observable. Let $\mathcal{P} = \mathcal{P}' > 0$ be the unique solution of the algebraic Riccati equation

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}' - 2\tau \mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} + I_{\bar{n}p} = 0. \tag{9}$$

We construct the observer

$$\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\hat{\chi}_i + \mathcal{B}_i u_i + S(\varepsilon)\mathcal{P}\mathcal{C}'(\zeta_i - \hat{\zeta}_i), \tag{10a}$$

$$\hat{\bar{x}}_i = (T_i' T_i)^{-1} T_i' \hat{\chi}_i, \tag{10b}$$

where $S(\varepsilon) = \text{blkdiag}(I_p \varepsilon^{-1}, \ldots, I_p \varepsilon^{-\bar{n}})$. Based on the observer estimate, we define the quantity

$$\eta_i = \mathcal{C}\hat{\chi}_i + \mathcal{D}_i u_i$$

to be shared with the other agents via the network's communication infrastructure (as described in Section II) and the observer-based control law

$$u_i = \bar{F}_i \hat{\bar{x}}_i. \tag{11}$$

Together, the observers for agents $i \in \{1, \ldots, n\} \setminus K$ form a distributed observer parameterized by a common high-gain parameter ε. The following lemma, which is proven in Appendix I, shows that all the observation errors vanish asymptotically if ε is chosen sufficiently small.

Lemma 4: There exists an $\varepsilon^* \in (0, 1]$ such that, if ε is chosen such that $\varepsilon \in (0, \varepsilon^*]$, then for each $i \in \{1, \ldots, n\} \setminus K$, $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$.

C. Main Result

By using the observer-based control law (11) for each agent $i \in \{1, \ldots, n\} \setminus K$, we achieve output synchronization. The following theorem formalizes this result.

Theorem 1: There exists an $\varepsilon^* \in (0, 1]$ such that, if ε is chosen such that $\varepsilon \in (0, \varepsilon^*]$, then output synchronization is achieved.

Proof: Since the systems are linear, the result follows from Lemmas 3 and 4 and the separation principle. ■

IV. Remarks on the Design Procedure

As described in Section III, the purpose of Step 1 is to reduce the dimension of the model (2) by removing redundant modes that cannot be observed from e_i. Such modes exist if agent i and agent K share particular unforced solutions. Consider, for example, the case where $A_i = A_K$ and $C_i = C_K$. Then the states x_i and x_K cannot be individually observed from $e_i = y_i - y_K$, since there are infinitely many initial conditions that yield the unforced solution $e_i = 0$. If, on the other hand, we define the state $\bar{x}_i = x_i - x_K$, then we obtain the model $\dot{\bar{x}}_i = A_i \bar{x}_i + B_i u_i$, $e_i = C_i \bar{x}_i + D_i u_i$, which is observable. Indeed, it is easily verified that in our design procedure, identical agents yield $q_i = n_i = n_K$ and $r_i = 0$, and that $\Lambda_i = \Phi_i = I_{n_i}$ is a valid choice; thus, one obtains precisely $\bar{x}_i = x_i - x_K$. In the general case, Step 1 yields a model (3) that incorporates the difference between modes that are shared between agents i

and K, in addition to the modes from both agent i and K that are not shared.

Step 2 consists of solving an output regulation problem, where the matrices $\Pi_i \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i \in \mathbb{R}^{m_i \times r_i}$ must be found from the regulator equations (6). A special situation arises when $r_i = 0$, which implies that $\bar{A}_{i22}$, $\bar{A}_{i12}$, and $\bar{C}_{i2}$ are empty matrices. In this case, Π_i and Γ_i are also empty matrices, and the need to solve the regulator equations vanishes. This situation occurs, in particular, if agent i and agent K are identical.

In Step 3, we introduce a state transformation from $\bar{x}_i$ to χ_i, where all the χ_i's have the same dimension $p\bar{n}$. Since the dimension of $\bar{x}_i$ may be less than $p\bar{n}$, the transformation to χ_i may involve an over-parameterization. In this case, (8) is not the only possible dynamical model of χ_i, but it is always *one* of the possible representations. Consider, for example, the scalar model $\dot{\bar{x}}_i = -\bar{x}_i + u_i$, $e_i = \bar{x}_i$ together with $\bar{n} = 2$. Then $\chi_i = [\bar{x}_i, -\bar{x}_i]'$, and two possible dynamical models are

$$\dot{\chi}_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \chi_i + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_i, \quad e_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \chi_i,$$

and

$$\dot{\chi}_i = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \chi_i + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_i, \quad e_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \chi_i.$$

Whereas the first model matches (8), the second one does not.

V. Example

We illustrate the results from Section III on a network of ten agents. Agents 1 and 2 are composed as the cascade of a second-order oscillator and a single integrator:

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad D_i = 0.$$

Agents 3, 4, and 5 are double integrators:

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 6, 7, and 8 are single integrators:

$$A_i = 0, \quad B_i = 1, \quad C_i = 1, \quad D_i = 0.$$

Finally, agents 9 and 10 are second-order mass-spring-damper systems:

$$A_i = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

The communication topology of the network is given by the digraph depicted in Fig. 1, which contains multiple directed spanning trees. One of these is rooted at node 2, and we therefore choose $K = 2$ for our design. The real part of the eigenvalues of the matrix $\bar{G}$, constructed by removing row and column 2 from the Laplacian of the digraph in Fig. 1, are lower bounded by approximately 0.33. We assume that a bound $\tau = 0.3$ is known during the design process. We also assume that a bound $\bar{n} = 6$ on $n_i + n_2$, $i \in \{1, \ldots, 10\} \setminus 2$, is known. Following the design procedure in Section III-B, we set $u_2 = 0$ and proceed with Steps 1–3 for each of the other agents.

Step 1: For illustrative purposes, we give the details for agent 3. Computing O_3, we get

$$O_3 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \implies q_3 = 1, \; r_3 = 2.$$

We may choose $\Lambda_{3u} = [1, 0]'$ and $\Phi_{3u} = [1, 0, 0]'$, and hence we can set $\Lambda_3 = I_2$ and $\Phi_3 = I_3$. It follows that

$$\bar{x}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_3 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_2.$$

It can be confirmed that the dynamics of $\bar{x}_i$ with output e_i takes the form of (3) with

$$\bar{A}_{312} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{322} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{C}_{32} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Step 2: The regulator equations (6) are easily found to have the solution

$$\Pi_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

We select the matrix $F_3 = [-2 - 3]$ to place the poles of $A_3 + B_3 F_3$ at -1 and -2. Thus, we obtain the matrix $\bar{F}_3 = [-2, -3, -3, -1]$.

Step 3: In Step 3 we design the observer that allows the controller (11) to be implemented based on observer estimates. We obtain

$$\mathcal{A} = \begin{bmatrix} 0 & I_5 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},$$
$$\mathcal{L}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & -0.5 & 0 \end{bmatrix}, \quad \mathcal{B}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We construct the observer according to the procedure in Section III-B, with the high-gain parameter $\varepsilon = 0.3$.

We perform the same procedure for the other agents. For agent 1, we obtain $q_i = 3$ and $r_i = 0$; for agents 6, 7, and 8, we obtain $q_i = 1$ and $r_i = 2$; and for agents 9 and 10, we obtain $q_i = 0$ and $r_i = 3$. Fig. 2 shows the resulting simulated output of all ten agents.

VI. Concluding Remarks

The designs presented in this paper rely on a set of conditions about the agents and the network that are straightforward to verify. However, they are not all strictly necessary. Inspecting the proofs of our results we see, for example, that the condition on the invariant zeros in Assumption 1 is used

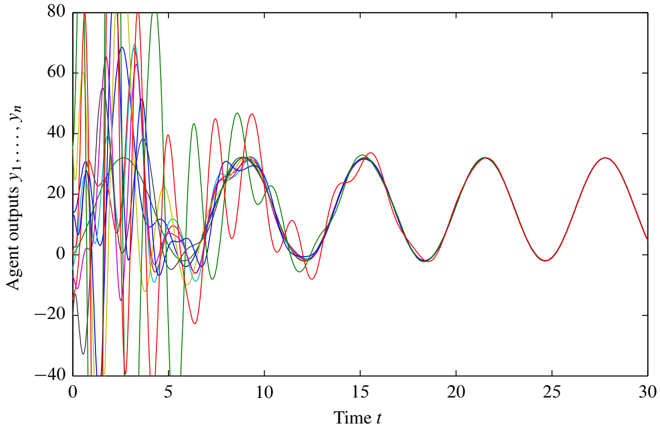

Fig. 2. Outputs from the simulation example

only in the proof of Lemma 3 to guarantee that no invariant zeros of (A_i, B_i, C_i, D_i) coincide with the eigenvalues of $\bar{A}_{i22}$. Since the eigenvalues of $\bar{A}_{i22}$ are only a subset of the eigenvalues of A_K, the quadruple (A_i, B_i, C_i, D_i) can be allowed to contain certain invariant zeros of A_K. Indeed, in the special case of identical agents, the matrix $\bar{A}_{i22}$ vanishes, so the condition on the invariant zeros is not needed.

Similarly, the condition of right-invertibility is used only to guarantee solvability of the regulator equations (6), which vanish for identical agents. Hence, if agent i is identical to agent K, then it does not need to be right-invertible. It is likely that, in the general case, the right-invertibility condition can be replaced with a relaxed invertibility condition for parts of the system. The development of such a condition is a topic of future research.

In this paper we have only been concerned with achieving output synchronization, without regard to the properties of the synchronization trajectory. Under conditions similar to Assumption 1 it is also possible to regulate the synchronization trajectory according to a reference model. The details of such a design will be included in an upcoming journal version of this paper.

Acknowledgements

The authors would like to thank Dr. Zhongkui Li for assistance regarding the proof of Lemma 5 of Li et al. [12].

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Appendix I
Proofs of Lemmas 1–4

Proof of Lemma 1: The set of nodes $\{1, \ldots, n\} \setminus K$ can be grouped into directed subgraphs $\mathscr{G}_1, \ldots, \mathscr{G}_M$, each of which has a directed spanning tree rooted at a child of node K. We can assume that there are no edges from $\mathscr{G}_k$ to $\mathscr{G}_j$ if $k > j$

(if such an edge exists, then the child node in $\mathcal{G}_j$ can be moved to $\mathcal{G}_k$). With this permutation, the matrix $\bar{G}$ takes the block-triangular form

$$\bar{G} = \begin{bmatrix} \bar{G}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \bar{G}_{M1} & \cdots & \bar{G}_M \end{bmatrix}.$$

Each submatrix $\bar{G}_i$, $i \in 1, \dots, M$, can be written as $\bar{G}_i = G_i + D_i$, where G_i is the Laplacian of $\mathcal{G}_i$ and D_i is a diagonal matrix whose j'th entry is the total weight of all the edges to node j of $\mathcal{G}_i$ from nodes outside of $\mathcal{G}_i$. Since $\mathcal{G}_i$ contains a directed spanning tree whose root is the child of node K, the diagonal element in D_i corresponding to that root is positive. It therefore follows from a version of Lemma 5 of Li et al. [12], given as Lemma 5 in Appendix II, that all the eigenvalues of $\bar{G}_i$ are in the open right-half complex plane. The same is true for $\bar{G}$, due to its block-triangular form. ■

Proof of Lemma 2: The definitions of Λ_{iu} and Φ_{iu} imply that the columns of $[\Lambda'_{iu}, \Phi'_{iu}]'$ span the unobservable subspace of the model (2), which is invariant with respect to $\text{blkdiag}(A_i, A_K)$. Hence, there exists a matrix $U_i \in \mathbb{R}^{q_i \times q_i}$ such that

$$\begin{aligned} \begin{bmatrix} A_i & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} &= \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} U_i, \\ \begin{bmatrix} C_i & -C_K \end{bmatrix} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} &= 0. \end{aligned} \tag{12}$$

Let $\bar{x}_i$ be partitioned as $\bar{x}_i = [\bar{x}'_{i1}, \bar{x}'_{i2}]'$, where

$$\bar{x}_{i1} = x_i - \Lambda_i M_i \Phi_i^{-1} x_K, \quad \bar{x}_{i2} = -N_i \Phi_i^{-1} x_K.$$

Using the equality $C_i \Lambda_{iu} = C_K \Phi_{iu}$, derived from (12), we calculate e_i in terms of $\bar{x}_{i1}$ and $\bar{x}_{i2}$:

$$\begin{aligned} e_i &= C_i x_i - C_K x_K + D_i u_i \\ &= C_i x_i - C_K \begin{bmatrix} \Phi_{iu} & \Phi_{io} \end{bmatrix} \Phi_i^{-1} x_K + D_i u_i \\ &= C_i x_i - \begin{bmatrix} C_i \Lambda_{iu} & C_K \Phi_{io} \end{bmatrix} \Phi_i^{-1} x_K + D_i u_i \\ &= C_i x_i - (C_i \Lambda_i M_i + C_K \Phi_i N'_i N_i) \Phi_i^{-1} x_K + D_i u_i \\ &= C_i (x_i - \Lambda_i M_i \Phi_i^{-1} x_K) - C_K \Phi_i N'_i N_i \Phi_i^{-1} x_K + D_i u_i \\ &= C_i \bar{x}_{i1} + C_K \Phi_i N'_i \bar{x}_{i2} + D_i u_i. \end{aligned}$$

From (12), we also have that $A_i \Lambda_{iu} = \Lambda_{iu} U_i$ and $A_K \Phi_{iu} = \Phi_{iu} U_i$. We therefore easily derive that there exist matrices Q_i and R_i on the form

$$Q_i = \begin{bmatrix} U_i & Q_{i12} \\ 0 & Q_{i22} \end{bmatrix}, \quad R_i = \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix},$$

such that $A_i \Lambda_i = \Lambda_i Q_i$ and $A_K \Phi_i = \Phi_i R_i$. For $\bar{x}_{i1}$ we can now calculate the state equations as

$$\begin{aligned} \dot{\bar{x}}_{i1} &= A_i x_i - \Lambda_i M_i \Phi_i^{-1} A_K x_K + B_i u_i \\ &= A_i x_i - \Lambda_i M_i R_i \Phi_i^{-1} x_K + B_i u_i \\ &= A_i x_i - \Lambda_i \begin{bmatrix} U_i & R_{i12} \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K + B_i u_i \\ &= A_i x_i - \Lambda_i \begin{bmatrix} U_i & 0 \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K \\ &\quad - \Lambda_i \begin{bmatrix} 0 & R_{i12} \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K + B_i u_i \\ &= A_i x_i - \Lambda_i Q_i M_i \Phi_i^{-1} x_K \\ &\quad - \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} N_i \Phi_i^{-1} x_K + B_i u_i \\ &= A_i \left(x_i - \Lambda_i M_i \Phi_i^{-1} x_K \right) \\ &\quad - \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} N_i \Phi_i^{-1} x_K + B_i u_i \\ &= A_i \bar{x}_{i1} + \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} \bar{x}_{i2} + B_i u_i. \end{aligned}$$

For $\bar{x}_{i2}$ we have

$$\begin{aligned} \dot{\bar{x}}_{i2} &= -N_i \Phi_i^{-1} A_K x_K = -N_i R_i \Phi_i^{-1} x_K \\ &= -R_{i22} N_i \Phi_i^{-1} x_K = R_{i22} \bar{x}_{i2}. \end{aligned}$$

Defining $\bar{A}_{i12} = \Lambda_i \left[\begin{smallmatrix} R_{i12} \\ 0 \end{smallmatrix} \right]$, $\bar{A}_{i22} = R_{i22}$, and $\bar{C}_{i2} = -C_K \Phi_i N'_i$, we see that e_i is governed by the dynamical equations (3). To see that $(\bar{A}_i, \bar{C}_i)$ is observable, note that the observability matrix O_i of the system (2) has rank $n_i + r_i$, which is precisely the order of the system (3). To see that the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K, note that, due to the block-triangular form of R_i, the eigenvalues of $\bar{A}_{i22} = R_{i22}$ are a subset of the eigenvalues of R_i. Since R_i is obtained via a similarity transform of A_K, $R_i = \Phi_i^{-1} A_K \Phi_i$, it has the same eigenvalues as A_K. ■

Proof of Lemma 3: Using the notation of the proof of Lemma 2, the task of achieving $\lim_{t \to \infty} e_i = 0$ can be viewed as an output regulation problem, where the subsystem $\dot{\bar{x}}_{i2} = \bar{A}_{i22} \bar{x}_{i2}$ is the exosystem and $\dot{\bar{x}}_{i1} = A_i \bar{x}_{i1} + \bar{A}_{i12} \bar{x}_{i2} + B_i u_i$ is the system to be regulated to achieve $e_i = C_i \bar{x}_{i1} - \bar{C}_{i2} \bar{x}_{i2} + D_i u_i = 0$. Since (A_i, B_i) is stabilizable and the eigenvalues of $\bar{A}_{i22}$ are in the closed right-half plane, the state-feedback controller $u_i = \bar{F}_i \bar{x}_i$ solves the regulation problem, assuming the regulator equations (6) are solvable [25, Theorem 2.3.1]. The regulator equations are solvable if, for each λ that is an eigenvalue of $\bar{A}_{i22}$, the Rosenbrock system matrix $\left[\begin{smallmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{smallmatrix} \right]$ is of full rank $n_i + p$ [25, Corollary 2.5.1]. This matrix has normal rank $n_i + p$ due to right-invertibility [26, Property 3.1.6]. Since (A_i, B_i, C_i, D_i) has no invariant zeros coinciding with eigenvalues of A_K and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K, it follows that the rank of the Rosenbrock system matrix is equal to the normal rank for each λ that is an eigenvalue of $\bar{A}_{i22}$. ■

Proof of Lemma 4: Let $\tilde{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\dot{\tilde{\chi}}_i = (\mathcal{A} + \mathcal{L}_i) \tilde{\chi}_i - S(\varepsilon) \mathcal{P} \mathcal{C}' (\zeta_i - \hat{\zeta}_i).$$

Noting that for each $i \in \{1, \dots, n\}$, $\sum_{j=1}^n g_{ij} = 0$, we have

$$\begin{aligned} \zeta_i &= \sum_{j=1}^n g_{ij} y_j = \sum_{j=1}^n g_{ij} (y_j - y_K) \\ &= \sum_{j \in \{1, \dots, n\} \setminus K} g_{ij} e_j = \sum_{j \in \{1, \dots, n\} \setminus K} g_{ij} (\mathcal{C} \chi_j + \mathcal{D}_j u_j). \end{aligned}$$

Also, since $\eta_K = 0$, we have $\hat{\zeta}_i = \sum_{j\in\{1,\ldots,n\}\setminus K} g_{ij}(\mathcal{C}\hat{\chi}_j + \mathcal{D}_j u_j)$. It follows that

$$\dot{\tilde{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\tilde{\chi}_i - S(\varepsilon)\sum_{j\in\{1,\ldots,n\}\setminus K} g_{ij}\mathcal{P}\mathcal{C}'\mathcal{C}\tilde{\chi}_j.$$

Introducing the state transformation $\xi_i = \varepsilon^{-1}S^{-1}(\varepsilon)\tilde{\chi}_i$, it can be confirmed that

$$\varepsilon\dot{\xi}_i = (\mathcal{A} + \mathcal{L}_{i\varepsilon})\xi_i - \sum_{j\in\{1,\ldots,n\}\setminus K} g_{ij}\mathcal{P}\mathcal{C}'\mathcal{C}\xi_j,$$

where $\mathcal{L}_{i\varepsilon} = \begin{bmatrix} 0 \\ \varepsilon^{\bar{n}+1}L_i S(\varepsilon)\end{bmatrix}$. Define $\xi = [\xi_1', \ldots, \xi_{K-1}', \xi_{K+1}', \ldots, \xi_n']'$, and $\mathcal{L}_\varepsilon = \mathrm{blkdiag}(\mathcal{L}_{1\varepsilon}, \ldots, \mathcal{L}_{(K-1)\varepsilon}, \mathcal{L}_{(K+1)\varepsilon}, \ldots, \mathcal{L}_{n\varepsilon})$, and note that $\|\mathcal{L}_\varepsilon\| = O(\varepsilon)$. The overall dynamics of ξ is

$$\varepsilon\dot{\xi} = (I_{n-1}\otimes\mathcal{A} + \mathcal{L}_\varepsilon - \bar{G}\otimes(\mathcal{P}\mathcal{C}'\mathcal{C}))\xi.$$

Following the methodology of Wu and Chua [14], we define U such that $J = U^{-1}\bar{G}U$, where J is the Jordan form of $\bar{G}$. Then, using the matrix $U\otimes I_{p\bar{n}}$ to perform a similarity transform of the matrix $(I_{n-1}\otimes\mathcal{A} - \bar{G}\otimes(\mathcal{P}\mathcal{C}'\mathcal{C}))$, we obtain

$$\begin{aligned}(U^{-1}\otimes I_{p\bar{n}})(I_{n-1}\otimes\mathcal{A} - \bar{G}\otimes(\mathcal{P}\mathcal{C}'\mathcal{C}))(U\otimes I_{p\bar{n}})\\ = I_{n-1}\otimes\mathcal{A} - J\otimes(\mathcal{P}\mathcal{C}'\mathcal{C}).\end{aligned}$$

Since the resulting matrix is upper block-triangular, we see that $(I_{n-1}\otimes\mathcal{A} - \bar{G}\otimes(\mathcal{P}\mathcal{C}'\mathcal{C}))$ is Hurwitz if $\mathcal{A} - \lambda\mathcal{P}\mathcal{C}'\mathcal{C}$ is Hurwitz for each λ that is an eigenvalue of $\bar{G}$. To see that the latter holds we follow the results of Yang et al. [13], noting that from (9),

$$\begin{aligned}&(\mathcal{A} - \lambda\mathcal{P}\mathcal{C}'\mathcal{C})\mathcal{P} + \mathcal{P}(\mathcal{A} - \lambda\mathcal{P}\mathcal{C}'\mathcal{C})^*\\ &= \mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}' - 2\mathrm{Re}(\lambda)\mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P}\\ &= \mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}' - 2\tau\mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} - 2(\mathrm{Re}(\lambda) - \tau)\mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} \le -I_{p\bar{n}}.\end{aligned}$$

Let therefore $P = P' > 0$ be the solution of the Lyapunov equation $P(I_{n-1}\otimes\mathcal{A} - \bar{G}\otimes(\mathcal{P}\mathcal{C}'\mathcal{C})) + (I_{n-1}\otimes\mathcal{A} - \bar{G}\otimes(\mathcal{P}\mathcal{C}'\mathcal{C}))'P = -I_{p\bar{n}(n-1)}$, and define the Lyapunov function candidate $V = \varepsilon\xi'P\xi$. We then have

$$\dot{V} = -\|\xi\|^2 + 2\xi'P\mathcal{L}_\varepsilon\xi \le -(1 - 2\|P\mathcal{L}_\varepsilon\|)\|\xi\|^2.$$

Since $\|\mathcal{L}_\varepsilon\| = O(\varepsilon)$, $\dot{V}$ is negative definite for sufficiently small ε, which implies $\lim_{t\to\infty}\xi = 0$. This in turn implies that $\hat{\chi}_i$ converges to $\chi_i = T_i\bar{x}_i$, and hence $\hat{\bar{x}}_i$ converges to $(T_i'T_i)^{-1}T_i'T_i\bar{x}_i = \bar{x}_i$. ∎

Appendix II
A Useful Lemma

We here give a version of Lemma 5 of Li et al. [12] taylored to this paper.

Lemma 5: Suppose that $\mathcal{G}$ is a weighted digraph with n nodes, containing a directed spanning tree with root node $K \in \{1, \ldots, n\}$. Let G be the Laplacian of $\mathcal{G}$ and let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a diagonal matrix with non-negative elements. If $d_K > 0$, then all the eigenvalues of $\bar{G} := G + D$ are in the open right-half complex plane.

Proof: Let $\hat{\mathcal{G}}$ denote an expanded digraph constructed from $\mathcal{G}$ by adding a node 0 and edges from node 0 to node $i \in \{1, \ldots, n\}$ with weigth d_i, whenever $d_i > 0$. Then the Laplacian of $\hat{\mathcal{G}}$ is given by $\hat{G} = \begin{bmatrix} 0 & 0 \\ -d & \bar{G}\end{bmatrix}$, where $d = [d_1, \ldots, d_n]'$. Since $\mathcal{G}$ contains an edge from node 0 to node K, $\hat{\mathcal{G}}$ contains a directed spanning tree rooted at node 0. Hence, from Lemma 3.3 of Ren and Beard [27], $\hat{G}$ has a simple eigenvalue at the origin, and all the other eigenvalues are in the open right-half complex plane. Due to the block-triangular form of $\hat{G}$, its eigenvalues consist of the zero element $(1, 1)$ and the eigenvalues of $\bar{G}$. Hence, the eigenvalues of $\bar{G}$ must be in the open right-half complex plane. ∎