Observer Design in the Presence of Periodic Output Disturbances by Mixing of Past and Present Output Data

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***Abstract*— We consider the problem of observer design for systems with periodic disturbances in the system outputs. Assuming that the period T of the disturbance is known, we introduce the idea of mixing past and present output data to remove the disturbance, by defining a new output as the difference between the original output at time t and at time $t-T$. We determine the exact conditions under which the system is observable from the redefined output, and specify how to design an observer using regular pole-placement techniques and linear time-invariant analysis. We validate the design through simulations.**

I. Introduction

A common problem in observer design is the presence of disturbances in the system outputs. A typical observer works by replicating the dynamics of a physical system and using an output injection term to stabilize the estimates and to achieve the required convergence properties. When the outputs from the physical system are corrupted by disturbances, the output injection term introduces disturbances in the observer dynamics, which can lead to severe performance degradation.

In many cases, output disturbances have a non-white correlation profile, meaning that the current value of the disturbance and past values of the disturbance have a common component, at least in a statistical sense. In this paper we introduce the idea of *mixing* past and present output data in order to remove or reduce such common components. The logic behind this mixing can be explained as follows: suppose an output signal $y(t)$ contains an additive disturbance signal $d(t)$ that has a significant positive correlation with the delayed disturbance signal $d(t-T)$. Suppose furthermore that we define a new output as $\bar{y}(t) = y(t)-y(t-T)$; that is, by subtracting past output data from the present output data. The disturbance term in $\bar{y}(t)$ is then $\bar{d}(t) = d(t)-d(t-T)$. Logically, one should expect the intensity of the signal $\bar{d}(t)$ to be significantly lower than that of $d(t)$, because a common component in $d(t)$ and $d(t-T)$ has been canceled. If we can design an observer based on the newly defined output $\bar{y}(t)$, chances are therefore good that it will be less affected by the disturbance than an observer based on the original output. A crucial condition is of course that the system must be observable from $\bar{y}(t)$.

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We shall consider the particular case of a linear time-invariant (LTI) system with outputs corrupted by periodic disturbances with known period T. Periodic disturbances can be completely canceled by redefining the output as described above. We shall investigate the exact conditions under which observability is retained from the redefined output, and show that an observer can be designed using standard techniques, by treating the system with the redefined output as another LTI system.

A. Deliberate Time Delays in Control and Estimation

The mixing approach discussed above involves the introduction of deliberate time delays in the observer. The potential power of deliberate time delays in control and estimation has been demonstrated in several ways. One example is the use of time delays for approximation of derivatives (e.g., [1]–[3]). Another is for stabilization of unknown periodic orbits and set points [4]–[7]. For periodic references or disturbances to system equations, time delays have been used to improve performance through *repetitive control* (see, e.g., [8]). In repetitive control, an internal model of an arbitrary periodic signal is created by using a time delay, and this model is gradually developed to help cancel periodic disturbances or improve tracking. Common to the approaches in [1]–[8] is that they result in retarded or neutral time-lag systems that are difficult to analyze with respect to stability and performance.

Of more direct relevance to the results in this paper is the use of time delays to create continuous-time observers with finite convergence time. The underlying idea, as described in simple terms in [9], is that for LTI systems, past output data can be related to the current state by an algebraic relationship. By using past and present observer estimates from two separate observers, [9] shows that enough equations are obtained to uniquely identify the state of the system. An earlier example of continuous-time observers with finite convergence time is found in [10], where the main idea is to use multiple delayed outputs to form a set of equations that is uniquely solvable with respect to the current state.

II. Problem Formulation

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{1a}$$

$$y(t) = Cx(t) + Du(t) + d(t), \quad y, d \in \mathbb{R}^p, \tag{1b}$$

where $u(t)$ is a known, piecewise continuous input that is bounded on any finite interval, and $d(t)$ is an unknown disturbance term in the measurement signal that is bounded, piecewise continuous, and periodic with period T. We assume that the system is initialized at time $t = 0$. In the absence of the disturbance $d(t)$, a standard observer (see, e.g., [11]) consists of a copy of the original system, plus an output injection term:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t) - Du(t)). \quad (2)$$

By defining the error $\tilde{x}(t) = x(t) - \hat{x}(t)$, one obtains the error dynamics

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t).$$

If the gain L is chosen such that the matrix $(A - LC)$ is Hurwitz, exponential stability of the error dynamics is ensured. Such a gain is guaranteed to exist (and can be easily found) if the pair (C, A) is observable, a property that can be confirmed by checking whether the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is of full rank n.

In the presence of a disturbance $d(t)$, the standard approach would yield the error dynamics

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t) - Ld(t). \quad (3)$$

Hence, unless L is chosen as zero, which is possible only if A is Hurwitz, the disturbance influences the observer error and prevents convergence to the origin.

A. Existing Methods for Disturbance Rejection

Much literature on estimation is devoted to dealing with unknown disturbances to the system's dynamic equations. Among the available techniques are the use of unknown-input observers, which are capable of perfectly canceling disturbances under restrictive conditions (see, e.g., [12]–[14], [15, Ch. 7]), and high-gain observers that can in some cases suppress the effect of a disturbance by increasing the observer gain (see, e.g., [16], [15, Ch. 8, 9]). Neither of these approaches are applicable when the disturbance occurs in the output signal.

One possibility is to extend the system with a model of the disturbance, and to estimate the states of this model along with the original states. When the disturbance is periodic and composed of a finite number of sinusoids with known frequencies, it can be modeled as the output of a marginally stable, linear exosystem. In this case an observer can be designed for the extended LTI system, just as for any other LTI system, provided it is observable (see, e.g., [15, Ch. 12, 13]). In the case of a general periodic disturbance with a known period, a time delay can be used to create an internal model of the disturbance, as mentioned in Section I. One possible approach is therefore to design an observer by the normal procedure in Section II, and to use a repetitive control approach to estimate the disturbance and to cancel its effect. As mentioned in Section I, however, repetitive control results in retarded or neutral time-lag systems that, owing to their infinite-dimensional nature, can be difficult to stabilize and analyze.

III. Mixing Design

To create an observer that is not influenced by the periodic disturbance, we redefine the output of the system as $\bar{y}(t) = y(t) - y(t - T)$. Since the disturbance has the property $d(t) = d(t - T)$, the new output $\bar{y}(t)$ is not influenced by the disturbance.

Because $\bar{y}(t)$ is defined by using a time delay, the system (1a) with output $\bar{y}(t)$ is infinite-dimensional. Although analysis of infinite-dimensional systems is often complicated, we shall demonstrate that in the present case, we can simplify matters by analyzing the dynamics of a finite-dimensional LTI system whose behavior coincides with the system in question. The trick is to relate the delayed signal $y(t-T)$ to the current state $x(t)$, rather than to the past state $x(t - T)$. Solving the linear differential equation (1), we have the following relationship for all $t \geq T$ (see, e.g., [11]):

$$x(t - T) = \mathrm{e}^{-AT}x(t) + \int_t^{t-T} \mathrm{e}^{A(t-T-\tau)}Bu(\tau)\,\mathrm{d}\tau. \quad (4)$$

Define $u^*(t) = C\int_t^{t-T} \mathrm{e}^{A(t-T-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t - T)$. Then for all $t \geq T$, $y(t-T) = C\mathrm{e}^{-AT}x(t) + u^*(t) + d(t - T)$. Hence, for all $t \geq T$, the system (1) with the new output corresponds precisely to the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (5a)$$

$$\bar{y}(t) = C(I - \mathrm{e}^{-AT})x(t) + Du(t) - u^*(t). \quad (5b)$$

We emphasize that the LTI description (5) is valid for all $t \geq T$, irrespective of the input $u(t)$. The signal $u^*(t)$ appearing in (5) can be computed for use in observer design, as described in Section III-B.

From (5) it is clear that if the pair $(C(I - \mathrm{e}^{-AT}), A)$ is observable, then we can design an observer for (5), which is also an observer for (1). The next theorem states the precise conditions for observability of the pair $(C(I - \mathrm{e}^{-AT}), A)$ in terms of the properties of the original system (1).

Theorem 1: The pair $(C(I - \mathrm{e}^{-AT}), A)$ is observable if, and only if, the pair (C, A) is observable and A has no purely imaginary eigenvalues located at $\pm 2\pi k/Tj$, $k = 0, 1, 2, \ldots$

Proof: To check observability, we look at the observability matrix for the pair $(C(I - \mathrm{e}^{-AT}), A)$:

$$\mathcal{O} = \begin{bmatrix} C(I - \mathrm{e}^{-AT}) \\ C(I - \mathrm{e}^{-AT})A \\ \vdots \\ C(I - \mathrm{e}^{-AT})A^{n-1} \end{bmatrix}.$$

Since

$$I - \mathrm{e}^{-AT} = I - \sum_{k=0}^{\infty} \frac{(-T)^k}{k!}A^k = -\sum_{k=1}^{\infty} \frac{(-T)^k}{k!}A^k,$$

it is obvious that $(I-\mathrm{e}^{-AT})$ commutes with A. Hence, we can equivalently write

$$\mathcal{O}=\begin{bmatrix} C(I-\mathrm{e}^{-AT}) \\ CA(I-\mathrm{e}^{-AT}) \\ \vdots \\ CA^{n-1}(I-\mathrm{e}^{-AT}) \end{bmatrix}=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}(I-\mathrm{e}^{-AT}). \quad (6)$$

From (6), we see that $\mathrm{rank}(\mathcal{O})=n$ if, and only if, both the $np\times n$ matrix $[C^\mathsf{T},(CA)^\mathsf{T},\ldots,(CA^{n-1})^\mathsf{T}]^\mathsf{T}$ and the $n\times n$ matrix $(I-\mathrm{e}^{-AT})$ are of full rank n. The matrix $[C^\mathsf{T},(CA)^\mathsf{T},\ldots,(CA^{n-1})^\mathsf{T}]^\mathsf{T}$ is the observability matrix of the pair (C,A). Hence, a necessary and sufficient condition for $\mathrm{rank}(\mathcal{O})=n$ is that (C,A) is observable and $(I-\mathrm{e}^{-AT})$ is nonsingular.

To complete the proof, we show that nonsingularity of $(I-\mathrm{e}^{-AT})$ is equivalent to A having no eigenvalues at $\pm 2\pi k/Tj$, $k=0,1,2,\ldots$ Let $A=PJP^{-1}$, where J is the Jordan normal form of A. Then $\mathrm{e}^{-AT}=P\mathrm{e}^{-JT}P^{-1}$. Singularity of $(I-\mathrm{e}^{-AT})$ is then equivalent to the existence of a nonzero vector z such that $(I-P\mathrm{e}^{-JT}P^{-1})z=0$. This is equivalent to $(I-\mathrm{e}^{-JT})P^{-1}z=0$, which in turn is equivalent to $(I-\mathrm{e}^{-JT})$ being singular. We have

$$\mathrm{e}^{-JT}=\begin{bmatrix} \mathrm{e}^{-J_1T} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{-J_qT} \end{bmatrix},$$

where $J_1,\ldots,J_q$ are the Jordan blocks of of J, corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_q$ of A, repeated according to their geometric multiplicities. Each block e^{-J_iT}, $i=1,\ldots,q$, has the form

$$\begin{bmatrix} \mathrm{e}^{-\lambda_iT} & \frac{(-T)\mathrm{e}^{-\lambda_iT}}{1!} & \cdots & \frac{(-T)^{r_i-2}\mathrm{e}^{-\lambda_iT}}{(r_i-2)!} & \frac{(-T)^{r_i-1}\mathrm{e}^{-\lambda_iT}}{(r_i-1)!} \\ 0 & \mathrm{e}^{-\lambda_iT} & \cdots & \frac{(-T)^{r_i-3}\mathrm{e}^{-\lambda_iT}}{(r_i-3)!} & \frac{(-T)^{r_i-2}\mathrm{e}^{-\lambda_iT}}{(r_i-2)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{-\lambda_iT} & \frac{(-T)\mathrm{e}^{-\lambda_iT}}{1!} \\ 0 & 0 & \cdots & 0 & \mathrm{e}^{-\lambda_iT} \end{bmatrix},$$

where r_i is the size of the Jordan block J_i. It follows that $(I-\mathrm{e}^{-JT})$ is singular if, and only if, there exists $i\in 1,\ldots,q$, such that $\mathrm{e}^{-\lambda_iT}=1$, causing one or more columns of $I-\mathrm{e}^{-JT}$ to vanish. This is equivalent to $-\lambda_iT=\pm 2\pi kj$, which is equivalent to $\lambda_i=\pm 2\pi k/Tj$. Hence, $(I-\mathrm{e}^{-AT})$ is nonsingular if, and only if, A has no eigenvalues at $\pm 2\pi k/Tj$, $k=0,1,2,\ldots$ ■

Remark 1: It can easily be seen that the condition in Theorem 1 is least restrictive if T is the fundamental period of the disturbance $d(t)$.

Remark 2: In the particular case when $d(t)$ is constant, a value $T>0$ that cancels $d(t)$ in $\bar{y}(t)$ and that satisfies the condition in Theorem 1 can be found if, and only if, (C,A) is observable and A has no eigenvalues at the origin. To see this, observe that a constant signal is periodic with all periods $T>0$, and if A has one or more pairs of non-zero eigenvalues on the imaginary axis, $T>0$ can always be chosen so that none of these pairs coincide with an integer multiple of $\pm 2\pi/Tj$.

A. Observer

Having established necessary and sufficient conditions for observability, we are now ready to proceed with the observer design. Because the system is precisely described by (5) for all $t\geq T$, the observer design can be carried out by the normal procedure outlined in Section II. Some care is required, however, because of the use of time delays in the observer.

We assume that the observer is initialized at time $t=t_0\geq 0$. To implement the observer, we need the signal $y(t-T)$. This requires the implementation of a time delay, which must be initialized with an initial function on the time interval $[t_0-T,t_0)$. To separate between the actual delayed signal $y(t-T)$ and the output of the implemented time delay, we use the notation $y_\mathrm{d}(t-T)$ for the latter. We assume that the time delay is initialized with a bounded initial function. Hence, $y_\mathrm{d}(t-T)$ is bounded for all $t\in[t_0,t_0+T)$, and for all $t\geq t_0+T$, $y_\mathrm{d}(t-T)=y(t-T)$. We shall also need the signal $u^*(t)$. As will be evident in Section III-B, the computation of $u^*(t)$ also depends on signals delayed by T. Similar to $y_\mathrm{d}(t)$, we therefore introduce $u^*_\mathrm{d}(t)$, with the property that $u^*_\mathrm{d}(t)$ is bounded for all $t\in[t_0,t_0+T)$, and for all $t\geq t_0+T$, $u^*_\mathrm{d}(t)=u^*(t)$.

Following the observer design procedure outlined in Section II with respect to the LTI system (5), we obtain the following observer:

$$\begin{aligned}\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)+L\Big(&y(t)-y_\mathrm{d}(t-T)\\ &-C(I-\mathrm{e}^{-AT})\hat{x}(t)-Du(t)+u^*_\mathrm{d}(t)\Big). \quad (7)\end{aligned}$$

Define the disturbance estimate $\hat{d}(t)=y(t)-C\hat{x}(t)-Du(t)$ and the associated error $\tilde{d}(t)=d(t)-\hat{d}(t)$. We can now state the following theorem:

Theorem 2: Suppose that the conditions of Theorem 1 hold, and let L be chosen such that the matrix $(A-LC(I-\mathrm{e}^{-AT}))$ is Hurwitz. Then $\tilde{x}(t)$ and $\tilde{d}(t)$ are bounded for all $t\in[t_0,t_0+T]$, and there exist constants $K>0$ and $\lambda>0$ such that for all $t\geq t_0+T$, $\|\tilde{x}(t)\|\leq K\|\tilde{x}(t_0+T)\|\mathrm{e}^{-\lambda(t-t_0-T)}$ and $\|\tilde{d}(t)\|\leq K\|C\|\|\tilde{x}(t_0+T)\|\mathrm{e}^{-\lambda(t-t_0-T)}$.

Proof: From Theorem 1, the pair $(C(I-\mathrm{e}^{-AT}),A)$ is observable. Hence, we can choose L such that $(A-LC(I-\mathrm{e}^{-AT}))$ is Hurwitz, using standard pole-placement techniques. We first consider the behavior of the observer error for $t\geq t_0+T$. We then have $y(t)-y_\mathrm{d}(t-T)=\bar{y}(t)$, and $u^*_\mathrm{d}(t)=u^*(t)$. By using (5b), which is valid for all $t\geq t_0+T$, we can rewrite the observer equation (7) as

$$\dot{\hat{x}}(t)=A\hat{x}(t)+Bu(t)+LC(I-\mathrm{e}^{-AT})(x(t)-\hat{x}(t)).$$

By subtracting this expression from (5a), we obtain the error dynamics $\dot{\tilde{x}}(t)=(A-LC(I-\mathrm{e}^{-AT}))\tilde{x}(t)$, which is exponentially stable. Hence, there exist $K>0$ and $\lambda>0$ such that for all $t\geq t_0+T$, $\|\tilde{x}(t)\|\leq K\|\tilde{x}(t_0+T)\|\mathrm{e}^{-\lambda(t-t_0-T)}$. Since $\hat{d}(t)=y(t)-C\hat{x}(t)-Du(t)=d(t)+C\tilde{x}(t)$, we have $\tilde{d}(t)=-C\tilde{x}(t)$, and hence $\|\tilde{d}(t)\|\leq K\|C\|\|\tilde{x}(t_0+T)\|\mathrm{e}^{-\lambda(t-t_0-T)}$.

It still remains to show that $\tilde{x}(t)$ and $\tilde{d}(t)$ are bounded for all $t \in [t_0, t_0+T]$. Define $v(t) = Bu(t) + L(y(t) - y_{\mathrm{d}}(t-T) - Du(t) + u_{\mathrm{d}}^*(t))$. Then for all $t \in [t_0, t_0+T]$, we have $\dot{\hat{x}}(t) = (A - LC(I - \mathrm{e}^{-AT}))\hat{x}(t) + v(t)$, where $v(t)$ is bounded. This represents an expontially stable LTI system with a bounded disturbance, which generates a bounded response. Hence $\hat{x}(t)$ is bounded on $[t_0, t_0+T]$, and it follows that $\tilde{x}(t)$ and $\tilde{d}(t)$ are bounded on the same interval. ■

B. Obtaining $u_{\mathrm{d}}^*(t)$

To implement the observer (7), we need to have access to the signal $u_{\mathrm{d}}^*(t)$ described in the previous section. This signal should have the property that for all $t \in [t_0, t_0+T)$, $u_{\mathrm{d}}^*(t)$ is bounded, and for all $t \geq t_0 + T$, $u_{\mathrm{d}}^*(t) = u^*(t) = C\int_t^{t-T} \mathrm{e}^{A(t-T-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t-T)$. We may calculate $u_{\mathrm{d}}^*(t)$ as follows:

$$\begin{aligned}\dot{z}(t) &= Az(t) + Bu(t),\\ u_{\mathrm{d}}^*(t) &= C(z_{\mathrm{d}}(t-T) - \mathrm{e}^{-AT}z(t)) + Du_{\mathrm{d}}(t-T).\end{aligned}$$

The internal state $z(t)$ is initialized at time t_0. The quantities $z_{\mathrm{d}}(t-T)$ and $u_{\mathrm{d}}(t-T)$ are delayed versions of $z(t)$ and $u(t)$, respectively, initialized with bounded initial functions on $[t_0-T, t_0)$, similar to $y_{\mathrm{d}}(t-T)$. To see why the computation is valid, we note that, from (4), the following expression holds for all $t \geq t_0 + T$:

$$z(t-T) = \mathrm{e}^{-AT}z(t) + \int_t^{t-T} \mathrm{e}^{A(t-T-\tau)}Bu(\tau)\,\mathrm{d}\tau. \quad (8)$$

Hence, for all $t \geq t_0+T$, $u^*(t) = C(z(t-T) - \mathrm{e}^{-AT}z(t)) + Du(t-T)$, and it follows that $u_{\mathrm{d}}^*(t) = u^*(t)$. This holds for any A, irrespective of the initial condition $z(t_0)$, even when A has positive eigenvalues (meaning that (1) is an unstable system).

Clearly, if A does have positive eigenvalues, the internal state $z(t)$ becomes unstable. Technically this is fine, because the system still has well-defined solutions for all $t \geq t_0$. In reality, however, internal instability leads to numerical problems as $z(t)$ becomes large. To deal with this, the internal state $z(t)$ may be reset with regular intervals. Of course, (8) becomes invalid for one period after any reset. To ensure continuous access to $u_{\mathrm{d}}^*(t)$, it is therefore necessary to create two sets of internal states, given by the same expression as $z(t)$. The resets of these two systems can be staggered in time such that (8) is always valid for one of them.

IV. Discussion

The mixing design presented in the previous sections is based on redefining the output map of the system (1) to obtain a new system representation (5). Because (5) represents an LTI system, the possibilities for observer design are not restricted to the particular observer (7); the system is amenable to the full range of observer design techniques for LTI systems and associated performance measures and methods of analysis. This includes, for example, the Kalman filter (see, e.g., [17]).

It is natural to ask whether redefining the output map results in unreasonably strict observability conditions in Theorem 1. To provide a partial answer to this question, we consider the special case when the disturbance $d(t)$ is a constant. In this case, an alternative way of solving the problem is to extend the system state to include the disturbance (which has derivative zero), and to design an observer for this system. The extended system becomes

$$\begin{aligned}\begin{bmatrix}\dot{x}(t)\\ \dot{d}(t)\end{bmatrix} &= \begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ d(t)\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}u(t),\\ y(t) &= \begin{bmatrix}C & I\end{bmatrix}\begin{bmatrix}x(t)\\ d(t)\end{bmatrix} + Du(t).\end{aligned}$$

The observability matrix for this system is

$$\mathcal{O} = \begin{bmatrix} C & I\\ CA & 0\\ \vdots & \vdots\\ CA^{n+p-1} & 0\end{bmatrix}.$$

Using the Cayley-Hamilton theorem, it is easily seen that $\mathcal{O}$ has full rank if, and only if, $\begin{bmatrix}(CA)^{\mathsf{T}} & \cdots & (CA^n)^{\mathsf{T}}\end{bmatrix}^{\mathsf{T}}$ has full rank, which happens if, and only if, the pair (C, A) is observable and A is nonsingular. This condition is equivalent to the condition that (C, A) is observable and that A has no eigenvalues at the origin, which is precisely the same condition as in Theorem 1 (see Remark 2).

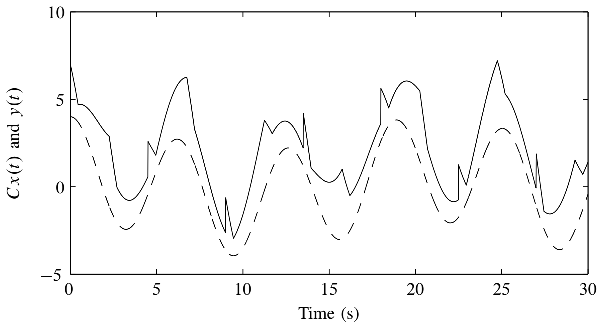
If the disturbance $d(t)$ is known, it is clearly trivial to design an observer for (1). A possible alternative solution is therefore to design the observer using the output $y(t)$ by the standard procedure in Section II, resulting in the error dynamics (3), and to subsequently identify $d(t)$ based on the available error signal $\tilde{y}(t) = y(t) - C\hat{x}(t) - Du(t) = C\tilde{x}(t) + d(t)$. If a stable left inverse exists for the system (3) with $d(t)$ considered the input and $\tilde{y}(t)$ considered the output, then $d(t)$ can be identified from $\tilde{y}(t)$. To have a stable left inverse, however, the system must be minimum-phase, and it turns out that this is the case only if the system matrix A is Hurwitz. To demonstrate this, we identify the zero dynamics of the system: setting $\tilde{y}(t) = 0$ and solving for $d(t)$ yields $d(t) = -C\tilde{x}(t)$. Inserting this into the error dynamics (3) yields $\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t) + LC\tilde{x}(t) = A\tilde{x}(t)$. The zero dynamics is therefore asymptotically stable only if A is Hurwitz.

The case when A is Hurwitz can be trivially solved by designing an observer with $L = 0$. Hence, the periodic output disturbance is primarily a problem when A is not Hurwitz, in which case the approach of the last paragraph cannot be used.

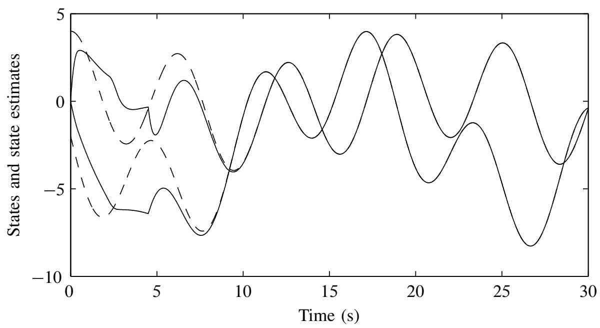
V. Simulation

To validate the mixing approach, we consider a marginally stable example system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x(t) + \begin{bmatrix}1\\ 0\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}x(t) + d(t)\end{aligned}$$



(a) The signal $Cx(t)$ (dashed) and the output $y(t)$ corrupted by a periodic signal (solid)

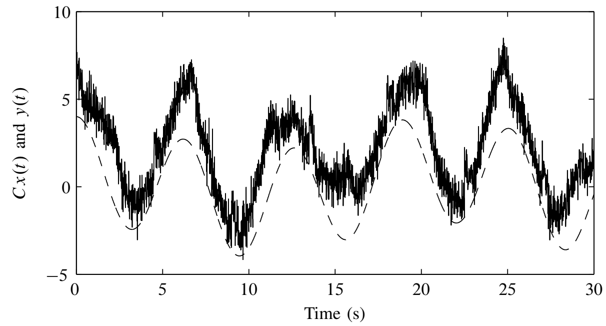


(b) States (dashed) and state estimates (solid)

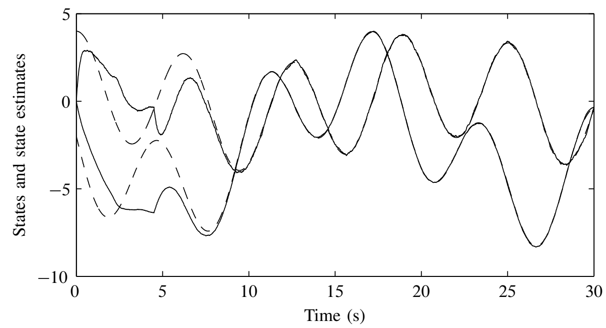
Fig. 1. Simulation results with periodic output disturbance

with $u(t) = 2+3\sin(0.3t)$ and $d(t)$ a signal with period $T = 4.5$ s. Figure 1(a) shows the disturbance-corrupted signal $y(t)$ together with $Cx(t)$. To check observability according to Theorem 1, we first note that the observability matrix of the pair (C, A) is the identity matrix; hence (C, A) is an observable pair. Second, we note that the eigenvalues of A are located at $\pm j$, which does not coincide with any integer multiple of $\pm 2\pi/Tj$. Hence, the system is observable from the output $\bar{y}(t) = y(t) - y(t-T)$, and we can implement the observer according to (7). We place the poles of $(A - LC(I - \mathrm{e}^{-AT}))$ at -1 and -2, which yields $L \approx \begin{bmatrix} 1.90 & -0.71 \end{bmatrix}^\mathsf{T}$. The result of the simulation is shown in Figure 1(b), where the actual states are compared to the state estimates.

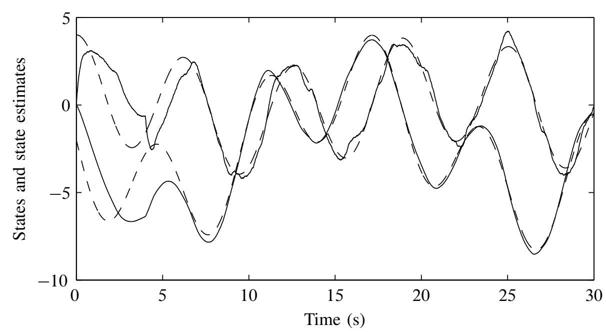
To investigate robustness to nonperiodic measurement noise, we add band-limited white noise to the output signal $y(t)$, as shown in Figure 2(a). The result of the simulation is shown in Figure 2(b). Finally, we investigate what happens if there is uncertainty in the period of the disturbance, in addition to measurement noise. We simulate the system with $T = 4$ s used in the observer implementation, which changes the observer gain to $L \approx \begin{bmatrix} 1.73 & -0.19 \end{bmatrix}^\mathsf{T}$ when the poles are placed as before. Figure 2(c) shows the states and the state estimates in this case. The result of using an incorrect period is to introduce a disturbance in the observer error dynamics. It is worth noting, however, that the observer cannot be destabilized by using an incorrect T. Further simulations confirm that the approach works with unstable system matrices as well.



(a) The signal $Cx(t)$ (dashed) and the output $y(t)$ (solid) corrupted by a periodic signal and band-limited white noise



(b) States (dashed) and state estimates (solid) with band-limited white measurement noise



(c) States (dashed) and state estimates (solid) with uncertain period and band-limited white measurement noise

Fig. 2. Simulation results with uncertain period and band-limited white measurement noise

VI. Concluding Remarks

We have presented a method for observer design in the presence of periodic disturbances in the system outputs, by redefining the output map to cancel the disturbance. As seen from (7), the observer depends on the system matrices A, B, C, and D, and thus the performance of the observer depends on the accuracy of the system model. This is not fundamentally different from standard observer designs, such as (2), which are also model-dependent. Nevertheless, the model-dependent computation of $u_\mathrm{d}^*(t)$ and the redefinition of the output map may increase sensitivity to modeling errors, in particular when the period T is large.

The simulation results shown in Figure 2(c) indicate that the design can produce acceptable results when the period T is uncertain; however, this is in large part due to the low gain used in the observer. When a higher gain is needed, for example, when dealing with unstable systems, the sensitivity to uncertainty in the period appears to be significant. Future research will focus on adaptation of the time delays used in the observer, in order to account for uncertainty or variation in the period.

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