Observers for Cascaded Nonlinear and Linear Systems

Håvard Fjær Grip, Ali Saberi, and Tor A. Johansen

Abstract— **In this paper we begin by assuming that an observer with a corresponding quadratic-type Lyapunov function has been designed for a given nonlinear system. We then consider the problem that arises when the output of that nonlinear system is not directly available; instead, it acts as an input to a second, linear system from which a partial-state measurement is in turn available. We develop an observer design methodology for the resulting cascade interconnection, based on estimating the unavailable output together with the states of the linear system. Under a set of technical assumptions, the overall error dynamics is proven to be globally exponentially stable if the gains are chosen to satisfy an H_∞ condition. We illustrate application of the methodology by considering the integration of inertial and satellite-based measurements.**

I. Introduction

Over the past decades, the topic of nonlinear state estimation has been extensively treated in the literature, and many different design methodologies have been developed. These include stochastic techniques such as the extended and unscented Kalman filter [1], [2]; the use of nonlinear state transformations to achieve linear error dynamics [3]; the use of linear observer dynamics in combination with a nonlinear transformation [4]; design of observer gains to achieve robustness against Lipschitz continuous nonlinearities [5]; the application of high gain to suppress Lipschitz continuous nonlinearities, both for left-invertible systems [6] and non-left-invertible systems [7]; the exploitation of monotonic nonlinearities [8]; sliding observers [9]; and moving-horizon estimation [10]. This list is by no means exhaustive, and in addition to general methodologies, application-specific designs proliferate throughout the literature.

In those cases where stability of the observation error can be explicitly proven, the main tool for doing so is often a Lyapunov function, and very often this Lyapunov function is of a *quadratic type*. Specifically, many continuous-time designs enable the construction of a Lyapunov function $V(t,\tilde{x})$ with the properties that $\alpha_1\|\tilde{x}\|^2 \le V(t,\tilde{x}) \le \alpha_2\|\tilde{x}\|^2$, $\dot{V}(t,\tilde{x}) \le -\alpha_3\|\tilde{x}\|^2$, and $\|[\partial V/\partial\tilde{x}](t,\tilde{x})\| \le \alpha_4\|\tilde{x}\|$, where $\tilde{x}$ is the error variable (e.g., [3]–[8]). Quadratic-type Lyapunov functions are also particularly well-suited for analyzing interconnected and singularly perturbed systems [11].

In this paper we assume that an observer with a corresponding quadratic-type Lyapunov function has *already* been designed for a given nonlinear system. We then consider the problem that arises when the output from that nonlinear system is not available directly, but instead available via a second, linear system. That is, the output from the nonlinear system acts as the input to a linear system, from which a partial-state measurement is in turn available. This situation results in a cascade interconnection that is illustrated in Fig. 1, and it is described by the system equations

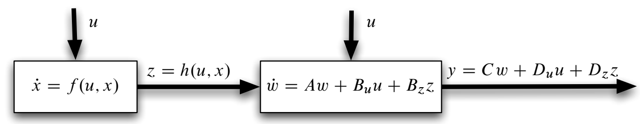

Fig. 1. Nonlinear and linear systems in cascade interconnection

$$\Sigma_1\colon \begin{cases} \dot{x} = f(u,x), \\ z = h(u,x), \end{cases} \tag{1a}$$

$$\Sigma_2\colon \begin{cases} \dot{w} = Aw + B_u u + B_z z, \\ y = Cw + D_u u + D_z z, \end{cases} \tag{1b}$$

where u is a vector of known time-varying signals, such as control inputs, reference signals, and measured disturbances. The partitioning of the system may stem from physical divisions or it may be a convenient choice for design purposes.

A. Relationship to Previous Work

The main idea behind our design is simple: since an observer is already available for the nonlinear Σ_1 subsystem with output z, we try to implement that observer using an estimate of z, denoted by $\hat{z}$. In order to produce such an estimate, we extend the state space of the linear subsystem Σ_2 to include z as an additional state, and construct and observer for this extended system.

The idea of extending the state space to obtain estimates of system inputs is not new. In particular, high-gain designs with an extended state space have been employed in recent years by Freidovich and Khalil [12], [13] for monitoring the decrease of Lyapunov functions and for transient performance recovery; and by the authors for the purpose of nonlinear parameter estimation [14], [15].

Our design methodology, being sequential in nature, is reminiscent of *recursive* observer design methodologies, where an observer is designed in stages for a chain of interconnected subsystems. We point in particular to the work of Shim and Seo [16], who treat the interconnection of a general nonlinear system—for which an observer with a corresponding quadratic-type Lyapunov function already

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The work of Håvard Fjær Grip is supported by the Research Council of Norway. The work of Ali Saberi is partially supported by NAVY grants ONR KKK777SB001 and ONR KKK760SB0012.

exists—with a second system consisting of MIMO chains of integrators with additive lower-triangular nonlinearities. Their problem formulation allows for nonlinearities in the second system that we do not permit in our Σ_2 subsystem; on the other hand, our Σ_2 subsystem covers a much wider range of *linear* systems. More importantly, the design of Shim and Seo [16] ensures stability through a fairly complicated multi-stage procedure that leaves little room for performance considerations, whereas our design only requires the construction of linear gains to ensure that the H_∞ norm of a particular transfer matrix is sufficiently small. This leaves the designer with a rich set of linear design tools.

B. Preliminaries

We denote by $\mathbb{R}_{\geq 0}$ the nonnegative real numbers. For a vector or matrix X, X' denotes its transpose. The operator $\|\cdot\|$ denotes the Euclidean norm for vectors and the Frobenius norm for matrices. For a symmetric positive-semidefinite matrix P, the minimum eigenvalue is denoted by $\lambda_{\min}(P)$. We assume that all signals are sufficiently smooth to allow differentiation when necessary. When considering systems of the form $\dot{\chi} = F(t, \chi)$, we assume that all functions involved are sufficiently smooth to guarantee that $F(t, \chi)$ is piecewise continuous in t and locally Lipschitz continuous in χ, uniformly in t, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. To simplify notation, we shall omit function arguments when possible.

II. Initial Assumptions

We consider the system (1), where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^{p_z}$, $w \in \mathbb{R}^{n_w}$, and $y \in \mathbb{R}^{p_y}$. Note that the system may have been transformed from its original coordinate basis in order to be better suited for observer design. We assume that an observer has already been constructed for the system Σ_1, on the form

$$\dot{\hat{x}} = f(u, \hat{x}) + g(u, \hat{x}, z). \tag{2}$$

This design could be implemented if z were available, yielding the error dynamics

$$\dot{\tilde{x}} = e(t, \tilde{x}), \tag{3}$$

where $\tilde{x} := x - \hat{x}$ and $e(t, \tilde{x}) := f(u(t), x(t)) - f(u(t), x(t) - \tilde{x}) - g(u(t), x(t) - \tilde{x}, z(t))$.

Assumption 1: There exists a continuously differentiable function $V\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and positive constants $\alpha_1, \ldots, \alpha_4$ such that for all $(t, \tilde{x}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x}$,

$$\alpha_1 \|\tilde{x}\|^2 \leq V(t, \tilde{x}) \leq \alpha_2 \|\tilde{x}\|^2, \tag{4}$$

$$\frac{\partial V}{\partial t}(t, \tilde{x}) + \frac{\partial V}{\partial \tilde{x}}(t, \tilde{x}) e(t, \tilde{x}) \leq -\alpha_3 \|\tilde{x}\|^2, \tag{5}$$

$$\left\| \frac{\partial V}{\partial \tilde{x}}(t, \tilde{x}) \right\| \leq \alpha_4 \|\tilde{x}\|. \tag{6}$$

Assumption 1 readily ensures exponential stability of the error dynamics (3) [17, Th. 4.10].

Because we will eventually replace z in (2) with an estimate $\hat{z}$, we need an assumption regarding the sensitivity of $g(u, \hat{x}, z)$ to changes in z.

Assumption 2: There exists an $L_1 > 0$ such that for all $(t, \hat{x}, \hat{z}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{p_z}$, $\|g(u(t), \hat{x}, z(t)) - g(u(t), \hat{x}, \hat{z})\| \leq L_1 \|z(t) - \hat{z}\|$.

For our analysis it is convenient to define a function

$$d(u, \dot{u}, x) = \frac{\partial h}{\partial u}(u, x)\dot{u} + \frac{\partial h}{\partial x}(u, x) f(u, x), \tag{7}$$

corresponding to the time derivative of the output z.

Assumption 3: There exists an $L_2 > 0$ such that for all $(t, \hat{x}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x}$, $\|d(u(t), \dot{u}(t), x(t)) - d(u(t), \dot{u}(t), \hat{x})\| \leq L_2 \|x(t) - \hat{x}\|$.

Remark 1: Assumptions 2 and 3 specify global Lipschitz-type conditions on the functions g and d. These conditions are clearly restrictive; however, they can be made far less restrictive if one assumes that $x(t)$ and $z(t)$ have known bounds and therefore belong to compact sets X and Z. This assumption is usually reasonable for physically motivated estimation problems. In particular, if $g(u, \hat{x}, z)$ is *locally* Lipschitz continuous in z, uniformly in $(u, \hat{x})$, then one can saturate the argument z in $g(u, \hat{x}, z)$ outside of Z to ensure that Assumption 2 holds. Similarly, the functions f and h can be redefined by arbitrarily extending them outside of X (e.g., through the use of saturations) to ensure that Assumption 3 holds. This strategy is common in the estimation literature (see, e.g., [7]).

Our final assumption specifies the requirements regarding the linear system Σ_2.

Assumption 4: The pair (A, C) is detectable; and the quadruple (A, B_z, C, D_z) is left-invertible with no invariant zeros at the origin.

Remark 2: Left-invertibility of a linear system means that two trajectories originating from the same initial condition will produce identical outputs for all $t \geq 0$ only if the inputs are also identical for all $t \geq 0$ [18, Ch. 3.2.2]. For example, every SISO system is left-invertible (unless its transfer function is identically zero).

Remark 3: We also assume, without loss of generality, that the matrices $[B_z', D_z']'$ and $[C, D_z]$ are of maximal rank p_z and p_y, respectively (i.e., there are no redundant elements of z and the elements of y are linearly independent). If this assumption does not hold, it is easily satisfied by redefining z or y to eliminate redundancies.

III. Observer Design

To introduce our methodology, we start by introducing an extended version of the linear system Σ_2, which includes z as an additional state. The extended system vector is given by $w_e = [w', z']'$, with dynamics

$$\dot{w}_e = \mathcal{A} w_e + \mathcal{B}_u u + \mathcal{B}_d d(u, \dot{u}, x), \tag{8a}$$

$$y = \mathcal{C} w_e + D_u u, \tag{8b}$$

where

$$\mathcal{A} = \begin{bmatrix} A & B_z \\ 0 & 0 \end{bmatrix}, \ \mathcal{B}_u = \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \ \mathcal{B}_d = \begin{bmatrix} 0 \\ I \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} C & D_z \end{bmatrix}.$$

We define an observer for the extended system in the following way:

$$\dot{\hat{w}} = A\hat{w} + B_u u + B_z \hat{z}$$

$$
\begin{aligned}
&+ K_w(y - C\hat{w} - D_u u - D_z \hat{z}), && \text{(9a)}\\
\dot{\hat{\xi}} &= -\frac{\partial h}{\partial x}(u, \hat{x}) g(u, \hat{x}, \hat{z}) \\
&\quad + K_z(y - C\hat{w} - D_u u - D_z \hat{z}), && \text{(9b)}\\
\hat{z} &= h(u, \hat{x}) + \xi, && \text{(9c)}
\end{aligned}
$$

where K_w and K_z are observer gains to be determined. The variables $\hat{w}$ and $\hat{z}$ are estimates of w and z, and they are gathered in a vector $\hat{w}_e = [\hat{w}', \hat{z}']'$. It is convenient to analyze the observer with respect $\hat{w}_e$, which constitutes a nonsingular transformation from the original observer states $(\hat{w}, \xi)$.

In addition to the observer (9), we implement the already existing observer for the system Σ_1; however, we do so not based on z, but the estimate $\hat{z}$:

$$\dot{\hat{x}} = f(u, \hat{x}) + g(u, \hat{x}, \hat{z}). \tag{10}$$

By differentiating $\hat{z}$ it is now easily verified that the dynamics of $\hat{w}_e$ is described by

$$
\begin{aligned}
\dot{\hat{w}}_e &= \mathcal{A}\hat{w}_e + \mathcal{B}_u u + \mathcal{B}_d d(u, \dot{u}, \hat{x}) \\
&\quad + K(y - \mathcal{C}\hat{w}_e - D_u u),
\end{aligned} \tag{11}
$$

where $K = [K_w', K_z']'$.

A. Stability

Based on the dynamics of the extended system (8) and the observer dynamics (11), we obtain the following dynamics for the observation error $\tilde{w}_e := w_e - \hat{w}_e$:

$$\dot{\tilde{w}}_e = (\mathcal{A} - K\mathcal{C})\tilde{w}_e + \mathcal{B}_d \tilde{d}(t, \tilde{x}), \tag{12}$$

where $\tilde{d}(t, \tilde{x}) := d(u(t), \dot{u}(t), x(t)) - d(u(t), \dot{u}(t), x(t) - \tilde{x})$. Furthermore, the dynamics of the error $\tilde{x}$ can be written as

$$\dot{\tilde{x}} = e(t, \tilde{x}) + g(u, \hat{x}, z) - g(u, \hat{x}, \hat{z}). \tag{13}$$

Our goal is now to choose an observer gain matrix K to stabilize the error dynamics (12), (13). Toward this end, we define

$$H(s) = (Is - \mathcal{A} + K\mathcal{C})^{-1}\mathcal{B}_d, \tag{14}$$

which is the transfer matrix from the input point of $\tilde{d}(t, \tilde{x})$ in (12) to the error $\tilde{w}_e$. Based on this transfer matrix, we can state the following result, which is proven in the Appendix.

Theorem 1: If K is chosen such that $\mathcal{A} - K\mathcal{C}$ is Hurwitz and $\|H(s)\|_\infty < \gamma := 4\alpha_3/(4L_2^2 + \alpha_4^2 L_1^2)$, then the error dynamics (12), (13) is globally exponentially stable.

The next question is whether there exists a gain that satisfies the conditions of Theorem 1.

Theorem 2: There exists a $\gamma^* > 0$ such that, for all $\gamma > \gamma^*$, K can be chosen such that $\mathcal{A} - K\mathcal{C}$ is Hurwitz and $\|H(s)\|_\infty < \gamma$. Furthermore, if the quadruple (A, B_z, C, D_z) is minimum-phase, then $\gamma^* = 0$.

According to Theorem 2, which is proven in the Appendix, we can design K such that $\|H(s)\|_\infty$ comes arbitrarily close to some lower limit $\gamma^* \geq 0$. In general, this lower limit may not be small enough to satisfy Theorem 1. However, if we impose an additional minimum-phase condition on Σ_2, then the lower limit is zero.

B. Gain Synthesis and Tuning

Although Theorem 1 enables us to compute an explicit numerical value of γ such that $\|H(s)\|_\infty < \gamma$ ensures stability, such a computation is likely to be conservative and lead to poor performance. It is therefore preferable in practice to tune the observer by starting with a large value of γ and decreasing it gradually until the desired stability and performance is achieved. As a practical matter, ensuring that $\mathcal{A} - K\mathcal{C}$ is Hurwitz and that $\|H(s)\|_\infty < \gamma$ can be achieved using several different H_∞ design methods; specifically, Riccati-based methods, direct methods, and LMI-based methods [18].

The use of LMIs is attractive, because it allows for easy incorporation of additional performance criteria in the design process. For a given γ, it follows from the bounded-real lemma [18, Th. 11.45] that $\|H(s)\|_\infty < \gamma$ is satisfied by choosing $K = P^{-1}X$, where X and $P = P' > 0$ are solutions of the LMI

$$\begin{bmatrix} P\mathcal{A} + \mathcal{A}'P - X\mathcal{C} - \mathcal{C}'X' + I & P\mathcal{B}_d \\ \mathcal{B}_d' P & -\gamma^2 I \end{bmatrix} < 0. \tag{15}$$

The solution of this LMI is far from unique—there are additional degrees of freedom in choosing K that can be used to improve performance. In particular, it was shown by Chilali and Gahinet [19] that by including additional LMIs based on a common Lyapunov matrix P, it is possible to constrain the closed-loop poles to some convex LMI region (assuming the region is feasible for the given H_∞ objective), or to incorporate additional H_∞ or H_2 minimization objectives.

Of particular concern when designing observers is the effect of measurement noise, which is amplified by large gains. Using LMIs, we can incorporate an additional objective to limit the effect of measurement noise. Suppose that y is affected by additive noise n; that is, $y = Cw + D_u u + D_z z + Nn$. The transfer matrix from the input point of n to $\tilde{w}_e$ is then $G(s) := -(sI - \mathcal{A} + K\mathcal{C})^{-1}KN$. We can limit the effect of the measurement noise on $\tilde{w}_e$ by minimizing a bound on $\|G(s)\|_\infty$, while at the same time ensuring that (15) is satisfied. This is done by minimizing a value $\gamma_n^2 > 0$ subject to the LMIs (15) and

$$\begin{bmatrix} P\mathcal{A} + \mathcal{A}'P - X\mathcal{C} - \mathcal{C}'X' + I & -XN \\ -N'X' & -\gamma_n^2 I \end{bmatrix} < 0. \tag{16}$$

As in other H_∞-based design problems, it may be beneficial to pre-scale the various input and output channels of the transfer function to the same order of magnitude (see, e.g., [20]). In most cases, one also has the freedom to adjust gains in the observer for the Σ_1 subsystem, which will affect the performance of the overall observer and the best choice of gain K in ways that are difficult to characterize precisely. In general, a certain amount of tuning based on trial and error is needed to find the best combination of gains.

C. Matrix Variables as States

In some cases, such as our navigation example in Section IV, it is convenient to express the whole state or part of the state of the Σ_1 subsystem as a matrix X, rather than a vector.

To cover this case, we simply define $x = \operatorname{vec} X$, where the vec operator stacks the columns of a matrix to form a vector. Our assumptions and the observer for w_e are then defined in terms of x.

IV. Example: Inertial and Satellite Integration

In this section we consider the problem of integrating measurements from satellite-based navigation systems such as GPS with inertial measurements. This problem, referred to as GNSS/INS integration, has been studied for several decades (see, e.g., [21]). Most solutions are based on the extended Kalman filter, but recently there has been an interest in constructing nonlinear observers with lower computational complexity and with global or semiglobal stability proofs.

Most of the effort on nonlinear navigation observers has been directed toward the problem of estimating the attitude. An extensive survey of attitude estimation methods is given by Crassidis, Markley, and Cheng [22]. Vik and Fossen [23] studied the GNSS/INS integration problem with the assumption that the attitude could be measured independently from the position and velocity, whereas Hua [24] constructed algorithms based only on GNSS position and velocity together with inertial and magnetometer measurements. We consider the same problem as Hua [24], within the theoretical framework established in this paper.

Our goal is to illustrate the general methodology of this paper in a simple manner, and we therefore ignore some important aspects such as bias estimation and the effect of various noise sources. It is nonetheless easy to extend the design in several ways, as discussed briefly in Section IV-E. These extensions will serve as the basis for a complete design in a forthcoming application paper.

A. System Description

The dynamics of the system is described by the following equations:

$$\dot{R} = RS(\omega^{\mathrm{b}}), \tag{17a}$$

$$\dot{p}^{\mathrm{n}} = v^{\mathrm{n}}, \tag{17b}$$

$$\dot{v}^{\mathrm{n}} = a^{\mathrm{n}} + g^{\mathrm{n}}, \tag{17c}$$

where $R \in \mathrm{SO}(3)$ is a rotation matrix from the body-fixed coordinate system to an earth-fixed reference frame, which describes the attitude; p^{n} and v^{n} are the position and velocity in the earth-fixed frame; ω^{b} is the angular velocity of the body-fixed frame with respect to the earth-fixed frame, given in body-fixed coordinates; g^{n} is the gravity vector; and a^{n} is the proper acceleration in earth-fixed coordinates.[1] The function $S(\cdot)$ generates a skew-symmetric matrix from its argument, so that for any $x, y \in \mathbb{R}^3$, $S(x)y = x \times y$. We assume that p^{n} and v^{n} are available as measurements from the GNSS receiver. The inertial sensors provide measurements of ω^{b}, as well as an accelerometer measurement a^{b}, which is related to a^{n} by $a^{\mathrm{n}} = Ra^{\mathrm{b}}$. We furthermore assume that a magnetometer measurement m^{b} is available, and that the earth's magnetic field m^{n} at the corresponding location is known.

[1] In this model, the earth-fixed coordinate system is considered to be inertial. For high-precision applications, the earth's rotation must also be accounted for in the model.

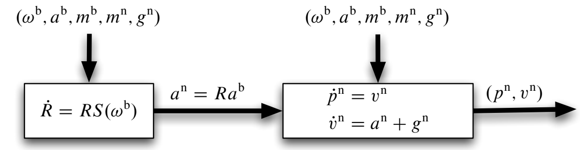

Fig. 2. Structure of system containing attitude, position, and velocity

B. Attitude Observer

Let us first consider the problem of estimating only the attitude R, assuming for the time being that a^{n} is available as a measurement. We construct an observer as

$$\dot{\hat{R}} = \hat{R}S(\omega^{\mathrm{b}}) + \Gamma J(m^{\mathrm{b}}, m^{\mathrm{n}}, a^{\mathrm{b}}, a^{\mathrm{n}}, \hat{R}), \tag{18}$$

where Γ is a symmetric positive-definite gain matrix and J is defined as

$$\begin{aligned} J &= A_{\mathrm{n}}A_{\mathrm{b}}' - \hat{R}A_{\mathrm{b}}A_{\mathrm{b}}', \\ A_{\mathrm{b}} &= \begin{bmatrix} m^{\mathrm{b}} & m^{\mathrm{b}} \times a^{\mathrm{b}} & m^{\mathrm{b}} \times (m^{\mathrm{b}} \times a^{\mathrm{b}}) \end{bmatrix}, \\ A_{\mathrm{n}} &= \begin{bmatrix} m^{\mathrm{n}} & m^{\mathrm{n}} \times a^{\mathrm{n}} & m^{\mathrm{n}} \times (m^{\mathrm{n}} \times a^{\mathrm{n}}) \end{bmatrix}. \end{aligned}$$

The definition of J is inspired by the TRIAD algorithm [25], which allows the attitude to be algebraically determined based on two body-fixed vector measurements and their corresponding reference vectors, provided the body-fixed vectors are non-parallel. To ensure that this is the case, we assume that there exists a constant $c_{\mathrm{obs}} > 0$ such that $\|m^{\mathrm{b}} \times a^{\mathrm{b}}\| \geq c_{\mathrm{obs}}$. We also assume that there is a positive constant $\underline{m}$ such that $\|m^{\mathrm{b}}\| \geq \underline{m}$, and that a^{b}, $\dot{a}^{\mathrm{b}}$, m^{b}, and ω^{b} are uniformly bounded.

The dynamics of the estimation error $\tilde{R} = R - \hat{R}$ is

$$\dot{\tilde{R}} = \tilde{R}S(\omega^{\mathrm{b}}) - \Gamma J(m^{\mathrm{b}}, m^{\mathrm{n}}, a^{\mathrm{b}}, a^{\mathrm{n}}, R - \tilde{R}). \tag{19}$$

Lemma 1: If Γ is chosen such that $\lambda_{\min}(\Gamma)$ is sufficiently large, then the error dynamics (19) is globally exponentially stable.

The proof of Lemma 1 is found in the Appendix.

C. GNSS/INS Integration

The observer (18) cannot be implemented, because it depends on the variable a^{n}, which is not measured. Instead, a^{n} acts as an input to the linear system (17b), (17c). This situation, which is illustrated in Fig. 2, corresponds to the problem formulation studied in this paper by defining the Σ_1 subsystem to contain the attitude and the Σ_2 subsystem to contain the position and velocity. In particular, let $x = \operatorname{vec} R$ (see Section III-C), $z = a^{\mathrm{n}} = Ra^{\mathrm{b}}$, $w = [p^{\mathrm{n}\prime}, v^{\mathrm{n}\prime}]'$, $y = [p^{\mathrm{n}\prime}, v^{\mathrm{n}\prime}]'$, and $u = [\omega^{\mathrm{b}\prime}, a^{\mathrm{b}\prime}, m^{\mathrm{b}\prime}, m^{\mathrm{n}\prime}, g^{\mathrm{n}}]'$. The following lemma is proven in the Appendix.

Lemma 2: If Γ is chosen such that $\lambda_{\min}(\Gamma)$ is sufficiently large, then Assumptions 1–3 are satisfied by the observer (18).

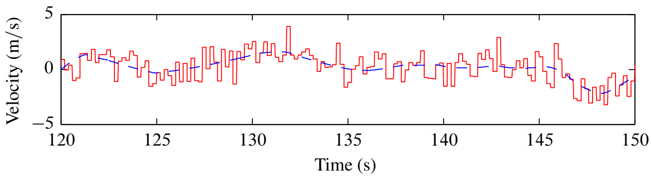

Fig. 3. Excerpt of true (blue, dashed) velocity and GNSS measurement (red, solid)

The matrices A, B_z, C, and D_z of Σ_2 are given by

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad D_z = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

It is trivial to verify that Assumption 4 holds for these matrices, and furthermore, that the quadruple (A, B_z, C, D_z) is minimum-phase. We can therefore apply our design methodology, which results in the following complete observer:

$$\dot{\hat{p}}^{\mathrm{n}} = \hat{v}^{\mathrm{n}} + k_{pp}(p^{\mathrm{n}} - \hat{p}^{\mathrm{n}}) + k_{pv}(v^{\mathrm{n}} - \hat{v}^{\mathrm{n}}), \tag{20a}$$

$$\begin{aligned} \dot{\hat{v}}^{\mathrm{n}} &= \hat{R}a^{\mathrm{b}} + \xi + g^{\mathrm{n}} \\ &\quad + k_{vp}(p^{\mathrm{n}} - \hat{p}^{\mathrm{n}}) + k_{vv}(v^{\mathrm{n}} - \hat{v}^{\mathrm{n}}), \end{aligned} \tag{20b}$$

$$\begin{aligned} \dot{\xi} &= -\Gamma J(m^{\mathrm{b}}, m^{\mathrm{n}}, a^{\mathrm{b}}, \hat{R}a^{\mathrm{b}} + \xi, \hat{R})a^{\mathrm{b}} \\ &\quad + k_{zp}(p^{\mathrm{n}} - \hat{p}^{\mathrm{n}}) + k_{zv}(v^{\mathrm{n}} - \hat{v}^{\mathrm{n}}), \end{aligned} \tag{20c}$$

$$\dot{\hat{R}} = \hat{R}S(\omega^{\mathrm{b}}) + \Gamma J(m^{\mathrm{b}}, m^{\mathrm{n}}, a^{\mathrm{b}}, \hat{R}a^{\mathrm{b}} + \xi, \hat{R}). \tag{20d}$$

The gain matrix K is made up of k_{pp}, k_{pv}, k_{vp}, k_{vv}, k_{zp}, and k_{zv}, and it must be chosen to ensure stability of the observer error dynamics. This is always possible, as stated in Theorem 3 below, which follows directly from our previous results.

Theorem 3: There exists a $\gamma > 0$ such that if K is chosen such that $\mathcal{A} - K\mathcal{C}$ is Hurwitz and $\|H(s)\|_\infty < \gamma$, then the error dynamics corresponding to the observer (20) is globally exponentially stable. Moreover, K can always be chosen to satisfy these conditions.

D. Simulation Results

In order to verify the design, we test it on a simulated takeoff, flight around a traffic pattern, and landing with a Cessna 172, using the *X-Plane*® flight simulator. The inertial measurements are available at a rate of 100 Hz, whereas the position and velocity measurements are available at a rate of 5 Hz. We have also added noise to the GNSS measurements. An example of true velocity versus the simulated GNSS measurement is shown in Fig. 3.

The observer is implemented with the gain for the attitude observer set to $\Gamma = \mathrm{diag}(20, 0.2, 0.2)$. This gain is chosen to emphasize the comparison between m^{n} and m^{b}, since these vectors are both available directly. To make the observer robust against the GNSS measurement errors, we follow the LMI-based design strategy described in Section III-B, by minimizing γ_n^2 subject to (15), (16) for some sufficiently small γ, with N selected as $N = I$. We find that we achieve stable estimates by choosing $\gamma = 50$, which yields the gains $k_{pp} \approx 128.9I$, $k_{pv} \approx 17.5I$, $k_{vp} \approx 15.7I$, $k_{vv} \approx 2.4I$, $k_{zp} \approx 1.3I$, and $k_{zv} \approx 0.2I$.

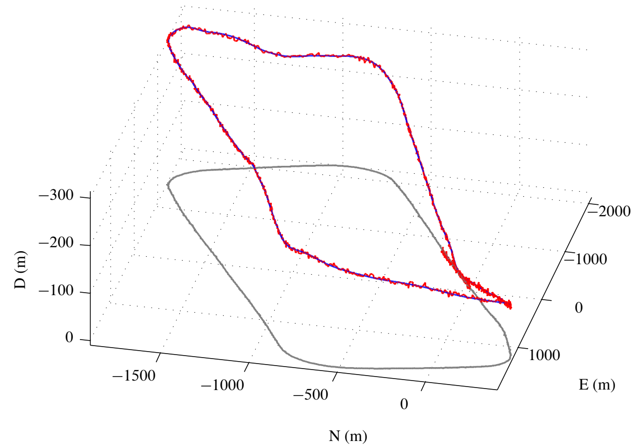

Fig. 4. True (blue, dashed) and estimated (red, solid) position in local North-East-Down coordinates (ground track at zero altitude shown in gray)

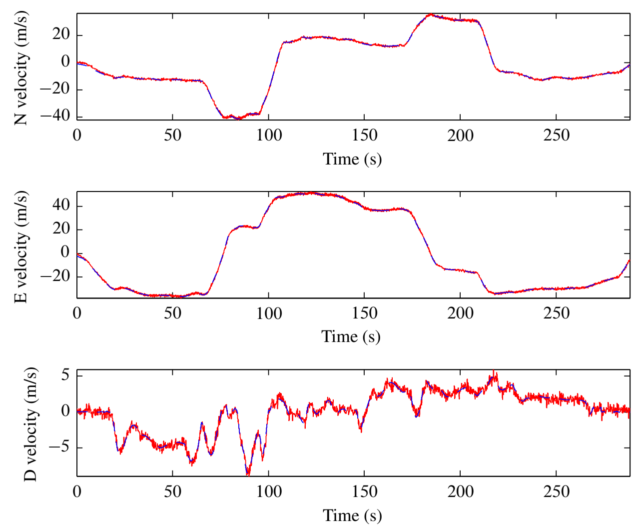

Fig. 5. True (blue, dashed) and estimated (red, solid) velocity in local North-East-Down coordinates

Figs. 4 and 5 show the true and estimated position and velocity of the vehicle. Fig. 6 shows the true Euler angles of the vehicle attitude, together with estimates derived from $\hat{R}$ by inverse trigonometry (see, e.g., [26]).

E. Extensions

One advantage of our design is that the observer can easily be adapted to changes in the linear part of the system. Suppose, for example, that the GNSS receiver only provides position measurements, which changes the measurement equation such that $C = [I, 0]$. Assumption 4 is still satisfied in this case, and (A, B_z, C, D_z) is still minimum-phase. Thus, we can apply our methodology with equal simplicity.

Another possibility is to include estimation of a constant accelerometer bias based on the bias estimator presented by Grip, Fossen, Johansen, and Saberi [27], [28]. Like the attitude observer, this bias estimator cannot be directly implemented because a^{n} is not available for measurement. However, it can be combined with (18) as an observer for

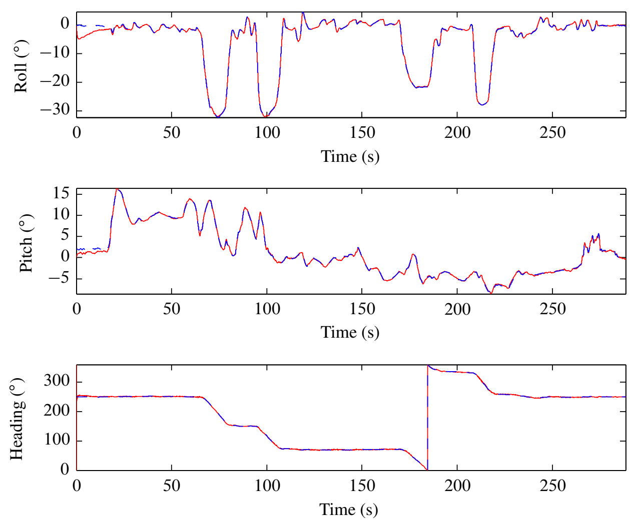

Fig. 6. True (blue, dashed) and estimated (red, solid) Euler angles

the Σ_1 subsystem. Further details of this design are omitted here due to space constraints. Adaptation of gyroscope bias as part of the observer for the Σ_1 subsystem is yet another possibility.

A disadvantage of our design is that $\hat{R}$ is in general not a rotation matrix, even though it converges to one. Thus, if one tries to extract the attitude in terms of Euler angles or quaternions from $\hat{R}$—as is frequently desirable—one might have an ill-defined problem, at least temporarily. An alternative is to use the convergent estimate of a^{n}, together with the other measurements, to estimate quaternions or a true rotation matrix, using any of a number of available methods (similar to one of the designs by Hua [24]).

V. Concluding Remarks

We have presented a design methodology that can be used to implement observers for certain nonlinear systems whose outputs are only indirectly available via a second, linear system. Future research will focus on application of the design methodology to a complete GNSS/INS integration design.

Appendix

Proof of Theorem 1: By the bounded-real lemma [18, Th. 11.45], the Hurwitz property of $\mathcal{A} - K\mathcal{C}$ and the H_∞ bound $\|H(s)\|_\infty < \gamma$ implies that the LMI (15) with $X = PK$ is satisfied for some positive definite P. Define the Lyapunov function $W(t, \tilde{x}, \tilde{w}_{\mathrm{e}}) = \gamma V(t, \tilde{x}) + \tilde{w}_{\mathrm{e}}' P \tilde{w}_{\mathrm{e}}$. Using Assumptions 1 and 2 we find that the derivative of W along the trajectories of (12), (13) satisfies

$$\begin{aligned}\dot{W} &\le -\gamma\alpha_3\|\tilde{x}\|^2 + \gamma\alpha_4 L_1\|\tilde{x}\|\|z - \hat{z}\| \\ &\quad + \tilde{w}_{\mathrm{e}}'(P\mathcal{A} + \mathcal{A}'P - X\mathcal{C} - \mathcal{C}'X')\tilde{w}_{\mathrm{e}} + 2\tilde{w}_{\mathrm{e}}' P \mathcal{B}_d \tilde{d} \\ &\le -\gamma\alpha_3\|\tilde{x}\|^2 + \gamma\alpha_4 L_1\|\tilde{x}\|\|\tilde{w}_{\mathrm{e}}\| - \|\tilde{w}_{\mathrm{e}}\|^2 + \gamma^2\|\tilde{d}\|^2 \\ &\quad + \begin{bmatrix}\tilde{w}_{\mathrm{e}} \\ \tilde{d}\end{bmatrix}' \begin{bmatrix} P\mathcal{A} + \mathcal{A}'P - X\mathcal{C} - \mathcal{C}'X' + I & P\mathcal{B}_d \\ \mathcal{B}_d' P & -\gamma^2 I\end{bmatrix} \begin{bmatrix}\tilde{w}_{\mathrm{e}} \\ \tilde{d}\end{bmatrix}.\end{aligned}$$

Using (15) and Assumption 3, we therefore have

$$\begin{aligned}\dot{W} &\le -\gamma\alpha_3\|\tilde{x}\|^2 + \gamma\alpha_4 L_1\|\tilde{x}\|\|\tilde{w}_{\mathrm{e}}\| - \|\tilde{w}_{\mathrm{e}}\|^2 + \gamma^2 L_2^2\|\tilde{x}\|^2 \\ &= -\begin{bmatrix}\|\tilde{x}\| \\ \|\tilde{w}_{\mathrm{e}}\|\end{bmatrix}' \begin{bmatrix}\gamma\alpha_3 - \gamma^2 L_2^2 & -\frac{1}{2}\gamma\alpha_4 L_1 \\ -\frac{1}{2}\gamma\alpha_4 L_1 & 1\end{bmatrix} \begin{bmatrix}\|\tilde{x}\| \\ \|\tilde{w}_{\mathrm{e}}\|\end{bmatrix}.\end{aligned}$$

The first- and second-order principal minors of the above matrix are positive if $\gamma < 4\alpha_3/(4L_2^2 + \alpha_4^2 L_1^2)$; thus the result follows [17, Th. 4.10]. ■

Proof of Theorem 2: We start by showing that the pair $(\mathcal{A}, \mathcal{C})$ is detectable. Consider any eigenvalue λ of $\mathcal{A}$ that is unobservable with respect to the pair $(\mathcal{A}, \mathcal{C})$. There exist $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{p_z}$, not both zero, such that

$$\begin{bmatrix}\mathcal{A} - \lambda I \\ \mathcal{C}\end{bmatrix}\begin{bmatrix} w \\ z\end{bmatrix} = 0 \iff \begin{bmatrix}(A - \lambda I)w + B_z z \\ -\lambda z \\ Cw + D_z z\end{bmatrix} = 0.$$

Clearly, either $z = 0$ or $\lambda = 0$. If $z = 0$, then it follows that $w \neq 0$ and moreover $\left[\begin{smallmatrix} A-\lambda I \\ C\end{smallmatrix}\right] w = 0$, which implies that λ is an unobservable eigenvalue of the pair (A, C). Since (A, C) is a detectable pair, λ must be in the open left-half complex plane. If $z \neq 0$, then $\lambda = 0$, and we have

$$\begin{bmatrix} A - \lambda I & B_z \\ C & D_z\end{bmatrix}\begin{bmatrix} w \\ z\end{bmatrix} = 0, \tag{21}$$

where left-hand side of (21) corresponds to the Rosenbrock system matrix for the quadruple (A, B_z, C, D_z). It follows that the Rosenbrock matrix has rank less than $n_w + p_z$ for $\lambda = 0$. The normal rank of the Rosenbrock matrix is $n_w + p_z$, which follows from left-invertibility according to [29, Property 3.1.6]. Hence, (A, B_z, C, D_z) has an invariant zero at the origin, which contradicts Assumption 4. Thus all unobservable eigenvalues of the pair $(\mathcal{A}, \mathcal{C})$ are in the open left-half complex plane.

Since the pair $(\mathcal{A}, \mathcal{C})$ is detectable, there exists a K such that $\mathcal{A} - K\mathcal{C}$ is Hurwitz, which implies that $\|H(s)\|_\infty < \bar{\gamma}$ for some $\bar{\gamma} > 0$. It follows that there exists a $\gamma^* \le \bar{\gamma}$ such that $\|H(s)\|_\infty < \gamma$ can be achieved for all $\gamma > \gamma^*$. This proves the first statement of the theorem.

To prove the second statement, we first show that since the quadruple (A, B_z, C, D_z) is left-invertible and minimum-phase, the same holds for the triple $(\mathcal{A}, \mathcal{B}_d, \mathcal{C})$. This triple is obtained from (A, B_z, C, D_z) by adding an integrator at each input point. Since integrators are left-invertible, it follows from the definition of left-invertibility that $(\mathcal{A}, \mathcal{B}_d, \mathcal{C})$ is left-invertible. Let λ be an invariant zero of $(\mathcal{A}, \mathcal{B}_d, \mathcal{C})$. Then the Rosenbrock system matrix corresponding to λ is rank deficient, so there exist $w \in \mathbb{R}^{n_w}$, $z \in \mathbb{R}^{p_z}$, and $d \in \mathbb{R}^{p_z}$, not all zero, such that

$$\begin{bmatrix}\mathcal{A} - \lambda I & \mathcal{B}_d \\ \mathcal{C} & 0\end{bmatrix}\begin{bmatrix} w \\ z \\ d\end{bmatrix} = 0 \iff \begin{bmatrix}(A - \lambda I)w + B_z z \\ -\lambda z + d \\ Cw + D_z z\end{bmatrix} = 0.$$

We must have $[w', z']' \neq 0$, for if this were not the case, we would have $d \neq 0$ and $z = 0$, which implies that $-\lambda z + d = d \neq 0$, which contradicts the above expression. From this expression we also see that (21) holds, which implies, by the same argument as above, that λ is an invariant zero of the quadruple (A, B_z, C, D_z). Hence λ is in the open left-half complex plane, which shows that $(\mathcal{A}, \mathcal{B}_d, \mathcal{C})$ is minimum-phase.

The error dynamics (12) is identical to the error dynamics obtained by constructing a strictly proper filter of the CSS architecture [18, eq. (9.12)], for a system $\dot{x} = \mathcal{A}x + \mathcal{B}_d u$, $y = \mathcal{C}x$, where x is the state to be estimated, u is an unknown input, and y is the available output. Hence, according to Saberi, Stoorvogel, and Sannuti [18, Th. 9.22], the gain matrix K can be chosen such that $\mathcal{A} - K\mathcal{C}$ is Hurwitz and $\|H(s)\|_\infty < \gamma$ for arbitrarily small $\gamma > 0$, provided the triple $(\mathcal{A}, \mathcal{B}_d, \mathcal{C})$ has no invariant zeros on the imaginary axis and the subspaces $\mathcal{S}^{-0}(\mathcal{A}, \mathcal{B}_d, \mathcal{C}, 0)$ and $\mathcal{V}^*(\mathcal{A}, \mathcal{B}_d, \mathcal{C}, 0)$, defined by Saberi et al. [18, Ch. 3.2.5], intersect only at the origin. These conditions hold as a consequence of $(\mathcal{A}, \mathcal{B}_d, \mathcal{C})$ being left-invertible and minimum-phase [18, Ch. 3.2.5], thus proving the second statement of the theorem. ■

Proof of Lemma 1: We have that $A_\mathrm{n} = RA_\mathrm{b}$, and we can therefore write $J = RA_\mathrm{b}A_\mathrm{b}' - \hat{R}A_\mathrm{b}A_\mathrm{b}' = \tilde{R}A_\mathrm{b}A_\mathrm{b}'$. Define the Lyapunov function $V(\tilde{R}) = \frac{1}{2}\|\tilde{R}\|^2$. We have

$$\begin{aligned}\dot{V} &= \mathrm{tr}(\tilde{R}'(\tilde{R}S(\omega^\mathrm{b}) - \Gamma\tilde{R}A_\mathrm{b}A_\mathrm{b}')) \\ &\le \sqrt{2}\|\tilde{R}\|^2\|\omega^\mathrm{b}\| - \|\Gamma^{1/2}\tilde{R}A_\mathrm{b}\|^2 \\ &\le \sqrt{2}\|\tilde{R}\|^2\|\omega^\mathrm{b}\| - \lambda_{\min}(A_\mathrm{b}A_\mathrm{b}')\lambda_{\min}(\Gamma)\|\tilde{R}\|^2.\end{aligned}$$

We can write $A_\mathrm{b} = Q\Lambda$, where the columns of Q are the normalized columns of A_b, and Λ is a diagonal matrix with elements corresponding to the norms of the columns of A_b. Since m^b and a^b are non-parallel (i.e., $\|m^\mathrm{b} \times a^\mathrm{b}\| \ge c_\mathrm{obs} > 0$), it follows from the definition of A_b that its columns are linearly independent and orthogonal; hence, the columns of Q are orthonormal. Thus $A_\mathrm{b}A_\mathrm{b}' = Q\Lambda^2Q'$ specifies a spectral decomposition of $A_\mathrm{b}A_\mathrm{b}'$, which implies that $\lambda_{\min}(A_\mathrm{b}A_\mathrm{b}')$ is equal to the smallest element of Λ^2; that is, $\lambda_{\min}(A_\mathrm{b}A_\mathrm{b}') = \min\{\|m^\mathrm{b}\|, \|m^\mathrm{b} \times a^\mathrm{b}\|, \|m^\mathrm{b} \times (m^\mathrm{b} \times a^\mathrm{b})\|\}^2 = c^2$, where $c = \min\{\underline{m}, c_\mathrm{obs}, \underline{m}c_\mathrm{obs}\} > 0$. Thus, $\dot{V} \le -(\lambda_{\min}(\Gamma)c^2 - \sqrt{2}\overline{\omega}^\mathrm{b})\|\tilde{R}\|^2$, where $\overline{\omega}^\mathrm{b}$ is a bound on $\|\omega^\mathrm{b}\|$. Clearly $\dot{V}$ is negative definite provided $\lambda_{\min}(\Gamma) > \sqrt{2}\overline{\omega}^\mathrm{b}/c^2$. The result now follows from application of the comparison lemma [17, Lemma 3.4]. ■

Proof of Lemma 2: Considering $V(\tilde{R})$ from the proof of Lemma 1 and noting that $V(\tilde{R}) = \frac{1}{2}\tilde{x}'\tilde{x}$, we see that (4) and (6) of Assumption 1 are satisfied. It also follows from the proof of Lemma 1 that (5) is satisfied. We have $g(u, \hat{x}, z) = \mathrm{vec}(\Gamma J(m^\mathrm{b}, m^\mathrm{n}, a^\mathrm{b}, a^\mathrm{n}, \hat{R}))$. It is therefore easy to confirm that $\|g(u, \hat{x}, z) - g(u, \hat{x}, \hat{z})\| = \|\Gamma[0, m^\mathrm{n} \times (z - \hat{z}), m^\mathrm{n} \times (m^\mathrm{n} \times (z - \hat{z}))]A_\mathrm{b}'\|$, and it follows from boundedness of m^n and A_b that Assumption 2 holds. Finally, we have $d(u, \dot{u}, x) = R\dot{a}^\mathrm{b} + RS(\omega^\mathrm{b})a^\mathrm{b}$, and hence $\|d(u, \dot{u}, x) - d(u, \dot{u}, \hat{x})\| = \|\tilde{R}(\dot{a}^\mathrm{b} + S(\omega^\mathrm{b})a^\mathrm{b})\|$. It follows from boundedness of ω^b, a^b, and $\dot{a}^\mathrm{b}$ that Assumption 3 holds. ■

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