Estimation of States and Parameters for Linear Systems with Nonlinearly Parameterized Perturbations

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Abstract

We consider systems that can be described by a linear part with a nonlinear perturbation, where the perturbation is parameterized by a vector of unknown, constant parameters. Under a set of technical assumptions about the perturbation and its relationship to the outputs, we present a modular design technique for estimating the system states and the unknown parameters. The design consists of a high-gain observer that estimates the states of the system together with the full perturbation, and a parameter estimator constructed by the designer to invert a nonlinear equation. We illustrate the technique on a simulated DC motor with friction.

Key words: estimation, nonlinear parameterization, observers

1. Introduction

A common problem in model-based control and estimation is the presence of uncertain perturbations. These perturbations may be the result of external disturbances or internal plant changes, such as a configuration change, system fault, or changes in plant characteristics. The uncertainty associated with the perturbations can in many cases be characterized by a vector of unknown, constant parameters.

Unknown parameters are often dealt with by introducing parameter estimates that are updated online in a suitable manner. Adaptive techniques for linearly parameterized systems have been treated in a large body of work (see, e.g., [1, 2]), but only a few specialized techniques have been developed for *nonlinearly* parameterized systems. Some of these methods are based on convexity

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or concavity of the parameterization [3, 4, 5]; these can also be extended to parameterizations that can be *convexified* through reparameterization (see [6]). Other methods apply to first-order systems with fractional parameterizations (e.g., [7, 8]). An approach that applies to higher-order systems with matrix fractional parameterizations—where an auxiliary estimate of the full perturbation is used in the estimation of the unknown parameters—was presented by Qu [9]. Tyukin, Prokhorov, and van Leeuwen [10] used the idea of *virtual algorithms*, which are designed as though the time derivative of the measurements were available, to construct a family of adaptation laws for monotonically parameterized perturbations. Other available methods include a hierarchical approach based on gridding the parameter space with a set of discrete candidate parameters [11] and a *feedback-domination* design utilizing a linearly parameterized bound on the nonlinearly parameterized terms [12]. In a recent paper, Liu, Ortega, Su, and Chu [13] presented a method based on *Immersion & Invariance* [14], where monotonic parameterizations are generated through the solution of a partial differential equation (see also [15]).

Grip, Johansen, Imsland, and Kaasa [16] have recently presented a design methodology for estimating unknown parameters in systems of the form $\dot{x} = f(t,x) + B(t,x)v(t,x) + \phi$, where $\phi = B(t,x)g(t,x,\theta)$ is a perturbation parameterized by the parameter vector θ. This methodology is based on the observation that, if the perturbation ϕ were directly available, then identifying θ would be a matter of inverting the nonlinear equation $\phi = B(t,x)g(t,x,\theta)$ with respect to θ. This line of thought leads to a modular design consisting of a *parameter estimator* and a *perturbation estimator*. The parameter estimator is designed as though ϕ were known, to dynamically invert the expression $\phi = B(t,x)g(t,x,\theta)$ with respect to θ. The perturbation estimator is designed to produce an estimate of ϕ to be used by the parameter estimator in lieu of the actual perturbation. The parameter estimate is in turn fed back to the perturbation estimator.

1.1. Topic of This Paper

Common to the techniques cited above, as well as additional results reviewed by Grip et al. [16], is that they are all based on the assumption of a full-state measurement. The goal of this paper is to extend the methodology of Grip et al. [16] to the case of a partial-state measurement.[1] In particular, we shall consider systems of the form $\dot{x} = Ax + Bu + E\phi$, $y = Cx$, and estimate both the state x and the unknown parameter vector θ. To achieve this, we replace the perturbation estimator by a high-gain observer for an *extended*

[1]A conference version of this paper has appeared as [17].

system. The state space of the extended system consists of the original state x together with the perturbation ϕ. The overall structure of the resulting estimator is illustrated in Figure 1.

The high-gain observer for the extended system is essentially a disturbance observer designed to estimate an unknown input, with a modification to make use of the parameter estimate. The strategy of estimating unknown perturbations for the purpose of control is not new; recently, it has been used to switch between controllers for alternative plant parameterizations by monitoring the decrease of a Lyapunov function [18]; and for the purpose of transient performance recovery [19, 20]. In their work [18, 19], Freidovich and Khalil employ a high-gain observer for an extended system that includes a perturbation appearing at the end of a single-input single-output integrator chain. Our high-gain design is closely related to their methodology; however, it encompasses a much larger class of multiple-input multiple-output systems.

The problem of estimating x together with θ can be viewed as a nonlinear observation problem, by defining θ as an augmented state with zero derivative. Depending on the specifics of the system, any one of a multitude of nonlinear observation methods—for example, extended Kalman filters, observers with linear error dynamics [2], high-gain observers [21, 22], and observers for systems with monotonic nonlinearities [23]—may therefore be used to estimate both states and parameters. Nevertheless, the property that the parameters are constant constitutes an important piece of structural information that justifies treating the simultaneous estimation of states and parameters as a special case and facilitates the modular design presented in this paper. We end by noting that Marino and Tomei [24] and Ding [25] have previously studied output-feedback control of nonlinearly parameterized systems; however, these designs do not involve precise estimation of the parameters.

1.2. Preliminaries

For a set of vectors $z_1, \dots, z_n$, we denote by $\mathrm{col}(z_1, \dots, z_n)$ the column vector obtained by stacking the elements of $z_1, \dots, z_n$. The operator $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced Euclidean norm for matrices. For a symmetric positive-semidefinite matrix P, the maximum and minimum eigenvalues are denoted $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$. For a set $E \subset \mathbb{R}^n$, we write $(E - E) := \{z_1 - z_2 \in \mathbb{R}^n \mid z_1, z_2 \in E\}$. When considering systems of the form $\dot{z} = F(t, z)$, we assume that all functions involved are sufficiently smooth to guarantee that $F\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz continuous in z, uniformly in t, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. The solution of this system, initialized at time $t = 0$ with initial condition $z(0)$ is denoted $z(t)$. We shall mostly simplify our notation by omitting function arguments.

2. Problem Formulation

We consider the following system:

$$\dot{x} = Ax + Bu + E\phi, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad \phi \in \mathbb{R}^k, \tag{1a}$$

$$y = Cx, \quad y \in \mathbb{R}^r, \tag{1b}$$

where x is the state; y is the output; ϕ is a perturbation to the system equations; and u is a time-varying input that is well-defined for all $t \in \mathbb{R}_{\geq 0}$ and may include control inputs, reference signals, measured disturbances, or other known influences. For ease of notation, we introduce the vector $v := \mathrm{col}(u, y)$ of known signals. The perturbation is given by the expression $\phi = g(v, x, \theta)$, where $g\colon \mathbb{R}^{m+r} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^k$ is continuously differentiable in its arguments, and $\theta \in \mathbb{R}^p$ is a vector of constant, unknown parameters. As we construct an estimator, additional smoothness requirements may be needed for g, as well as other functions, to guarantee that the piecewise continuity and local Lipschitz conditions in Section 1.2 hold for all systems involved.

To make the technicalities of our design easier to overcome, we make the following assumption:

Assumption 1. *The time derivative $\dot{u}$ is well-defined and piecewise continuous; there exist compact sets $X \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$, and $U' \subset \mathbb{R}^m$ such that for all $t \geq 0$, $x \in X$, $u \in U$, and $\dot{u} \in U'$; and there exists a known compact set $\bar{\Theta} \subset \mathbb{R}^p$ such that $\theta \in \bar{\Theta}$.*

Assumption 1 also implies that v and $\dot{v}$ belong to compact sets, which we denote by V and V'. In most estimation problems, it is reasonable to assume boundedness of the states and inputs, as they are typically derived from physical quantities with natural bounds.[2] When designing update laws for parameter estimates, we shall also assume that a parameter projection [1, App. E] can be implemented, restricting the parameter estimates to a compact, convex set $\Theta \supset \bar{\Theta}$.

We denote by $\Phi \subset \mathbb{R}^k$ the compact image of $V \times X \times \Theta$ under g. For ease of notation, we define

$$d(v, \dot{v}, x, \theta, \phi) := \frac{\partial g}{\partial v}(v, x, \theta)\dot{v} + \frac{\partial g}{\partial x}(v, x, \theta)(Ax + Bu + E\phi),$$

representing the time derivative of the perturbation ϕ.

[2] The assumption of boundedness does not imply any assumptions of stability regarding the model (1). It is only assumed that the trajectory of the physical system, generated by a specific initial condition and input, remains bounded.

Assumption 2. *The triple (C, A, E) is left-invertible and minimum-phase.*

Assumption 3. *There exists a number $\beta > 0$ such that for all $(v, \dot{v}, x, \theta, \phi) \in V \times V' \times X \times \Theta \times \Phi$ and for all $(\hat{x}, \hat{\theta}, \hat{\phi}) \in \mathbb{R}^n \times \Theta \times \mathbb{R}^k$, $\|d(v, \dot{v}, x, \theta, \phi) - d(v, \dot{v}, \hat{x}, \hat{\theta}, \hat{\phi})\| \leq \beta \|\mathrm{col}(x - \hat{x}, \theta - \hat{\theta}, \phi - \hat{\phi})\|$.*

Remark 1. *The Lipschitz-like condition in Assumption 3 is global in the sense that there are no bounds on $\hat{x}$ and $\hat{\phi}$. Although the condition may appear restrictive, we are free to redefine $g(v, x, \theta)$ for sufficiently large values of the arguments (i.e., outside the compact set $V \times X \times \Theta$) without altering the accuracy of the system description* (1)*. We may, for example, introduce a smooth saturation on x outside X, in which case the condition can be satisfied by requiring that $g(v, x, \theta)$ is sufficiently smooth. Similar techniques are frequently employed in the context of high-gain observers (see, e.g., [22]).*

In the following we first present the high-gain observer for the extended system and then recall the requirements placed on the parameter estimator.

3. High-Gain Observer for Extended System

When we include the perturbation ϕ as a state, we obtain an extended system

$$\begin{bmatrix} \dot{x} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I_k \end{bmatrix} d(v, \dot{v}, x, \theta, \phi), \tag{2}$$

with $y = Cx$. We implement an observer for this system of the following form:

$$\dot{\hat{x}} = A\hat{x} + Bu + E\hat{\phi} + K_x(\varepsilon)(y - C\hat{x}), \tag{3a}$$

$$\dot{z} = -\frac{\partial g}{\partial \theta}(v, \hat{x}, \hat{\theta})\dot{\hat{\theta}} - \frac{\partial g}{\partial x}(v, \hat{x}, \hat{\theta})K_x(\varepsilon)(y - C\hat{x}) + K_\phi(\varepsilon)(y - C\hat{x}), \tag{3b}$$

$$\hat{\phi} = g(v, \hat{x}, \hat{\theta}) + z, \tag{3c}$$

where $K_x(\varepsilon) \in \mathbb{R}^{n\times r}$ and $K_\phi(\varepsilon) \in \mathbb{R}^{k\times r}$ are gain matrices parameterized by a number $0 < \varepsilon \leq 1$ to be specified later. In (3), we have made use of a parameter estimate $\hat{\theta}$ and its time derivative. These values are produced by the parameter estimation module, which we shall discuss in Section 4. Taking the time derivative of $\hat{\phi}$ yields $\dot{\hat{\phi}} = d(v, \dot{v}, \hat{x}, \hat{\theta}, \hat{\phi}) + K_\phi(\varepsilon)(y - C\hat{x})$. Defining the errors $\tilde{x} = x - \hat{x}$, $\tilde{\phi} = \phi - \hat{\phi}$, and $\tilde{y} = y - C\hat{x}$, we may therefore write the error dynamics of the observer as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\phi}} \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ I_k \end{bmatrix} \tilde{d} - \begin{bmatrix} K_x(\varepsilon) \\ K_\phi(\varepsilon) \end{bmatrix} \tilde{y}, \tag{4}$$

where $\tilde{d} := d(v, \dot{v}, x, \theta, \phi) - d(v, \dot{v}, \hat{x}, \hat{\theta}, \hat{\phi})$. In the error dynamics (4), the term $\tilde{d}$ acts as an unknown disturbance, and $\tilde{y}$ is an available output. Our goal is to design a family of gains $K(\varepsilon) := [K_x^\top(\varepsilon), K_\phi^\top(\varepsilon)]^\top$ such that, as the number ε becomes small, the high-gain observer produces stable estimates with a diminishing effect from the parameter error $\tilde{\theta} := \theta - \hat{\theta}$. Before proceeding with the design, we need the following lemma.

Lemma 1. *The error dynamics* (4) *with input $\tilde{d}$ and output $\tilde{y}$, and gain $K(\varepsilon) = 0$, is left-invertible and minimum-phase.*

Proof. See Appendix A. □

Consider the error dynamics (4) with $K(\varepsilon) = 0$. Let n_a and n_q denote the number of invariant zeros and infinite zeros of the system, respectively, and define $n_b = n - n_a - n_q$. Let Λ_1, Λ_2, and Λ_3 denote the nonsingular state, output, and input transformations that take the system (4) with input $\tilde{d}$ and output $\tilde{y}$, and with $K(\varepsilon) = 0$, to the *special coordinate basis* (SCB) of Sannuti and Saberi [26]. We apply these transformations to the system (4) (without setting $K(\varepsilon) = 0$), by writing $\mathrm{col}(\tilde{x}, \tilde{\phi}) = \Lambda_1 \chi$, $\tilde{y} = \Lambda_2 y$, and $\tilde{d} = \Lambda_3 \delta$. From the work of Saberi and Sannuti [21, Th. 2.6], we can partition the transformed state as $\chi = \mathrm{col}(\chi_a, \chi_b, \chi_q)$, where χ_a has dimension n_a, χ_b has dimension n_b, and χ_q has dimension n_q. The variable χ_q is further partitioned as $\chi_q = \mathrm{col}(\chi_{q_1}, \dots, \chi_{q_k})$, where each χ_{q_j}, $j = 1, \dots, k$, has dimension n_{q_j}. The transformed output is partitioned as $y = \mathrm{col}(y_q, y_b)$, where y_q has dimension k and is further partitioned as $y_q = \mathrm{col}(y_{q_1}, \dots, y_{q_k})$, and y_b has dimension $r_b := r - k$. The transformed input has dimension k and is further partitioned as $\delta = \mathrm{col}(\delta_1, \dots, \delta_k)$. The resulting system is written as

$$\dot{\chi}_a = A_a \chi_a + L_{aq} y_q + L_{ab} y_b - K_{aq}(\varepsilon) y_q - K_{ab}(\varepsilon) y_b, \tag{5a}$$

$$\dot{\chi}_b = A_b \chi_b + L_{bq} y_q - K_{bq}(\varepsilon) y_q - K_{bb}(\varepsilon) y_b, \tag{5b}$$

$$\begin{aligned} \dot{\chi}_{q_j} = {} & A_{q_j} \chi_{q_j} + L_{q_j q} y_q - K_{q_j q}(\varepsilon) y_q - K_{q_j b}(\varepsilon) y_b \\ & + B_{q_j}(\delta_j + D_{a_j} \chi_a + D_{b_j} \chi_b + D_{q_j} \chi_q), \end{aligned} \tag{5c}$$

$$y_b = C_b \chi_b, \quad y_{q_j} = C_{q_j} \chi_{q_j}, \tag{5d}$$

where A_{q_j}, B_{q_j}, and C_{q_j} have the special structure

$$A_{q_j} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_{q_j} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_{q_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{6}$$

and where (C_b, A_b) is an observable pair. The gains $K_{aq}(\varepsilon)$, $K_{ab}(\varepsilon)$, $K_{bq}(\varepsilon)$, $K_{bb}(\varepsilon)$, $K_{q_jq}(\varepsilon)$, and $K_{q_jb}(\varepsilon)$, $j = 1, \dots, k$, are related to $K(\varepsilon)$ by

$$K(\varepsilon) = \Lambda_1 \begin{bmatrix} K_{aq}(\varepsilon) & K_{ab}(\varepsilon) \\ K_{bq}(\varepsilon) & K_{bb}(\varepsilon) \\ K_{qq}(\varepsilon) & K_{qb}(\varepsilon) \end{bmatrix} \Lambda_2^{-1}, \tag{7}$$

where $K_{qq}(\varepsilon) = [K_{q_1q}^{\mathsf{T}}(\varepsilon), \dots, K_{q_kq}^{\mathsf{T}}(\varepsilon)]^{\mathsf{T}}$ and $K_{qb}(\varepsilon) = [K_{q_1b}^{\mathsf{T}}(\varepsilon), \dots, K_{q_kb}^{\mathsf{T}}(\varepsilon)]^{\mathsf{T}}$. Once we have chosen $K_{aq}(\varepsilon)$, $K_{ab}(\varepsilon)$, $K_{bq}(\varepsilon)$, $K_{bb}(\varepsilon)$, $K_{qq}(\varepsilon)$, and $K_{qb}(\varepsilon)$, we can therefore implement the high-gain observer (3) for the extended system with the gains given by (7).

Remark 2. *In* (5), *the* χ_a *subsystem represents the zero dynamics, with the eigenvalues of* A_a *corresponding to the invariant zeros of the system. The* χ_b *subsystem represents states that are observable from the output* y_b, *but that are not directly affected by any inputs. The* χ_q *subsystem represents the infinite zero structure, and it consists of* k *integrator chains from scalar inputs* δ_j *to scalar outputs* y_{q_j}, *with interconnections to other subsystems at the lowest level of each integrator chain.*

3.1. Design of Gains

Let $K_{bb}(\varepsilon) = K_{bb}$ be chosen independently from ε such that the matrix $A_b - K_{bb}C_b$ is Hurwitz. This is always possible, since the pair (C_b, A_b) is observable. For each $j \in 1, \dots, k$, select $\bar{K}_{q_j} := \operatorname{col}(\bar{K}_{q_j1}, \dots, \bar{K}_{q_jn_{q_j}})$ such that the matrix $H_j := A_{q_j} - \bar{K}_{q_j}C_{q_j}$ is Hurwitz. Then, let $K_{aq}(\varepsilon) = L_{aq}$, $K_{ab}(\varepsilon) = L_{ab}$, $K_{bq}(\varepsilon) = L_{bq}$, $K_{qb}(\varepsilon) = 0$, and let

$$K_{q_jq}(\varepsilon) = \begin{bmatrix} 0_{n_{q_j}\times(j-1)} & \operatorname{col}(\bar{K}_{q_j1}/\varepsilon, \dots, \bar{K}_{q_jn_{q_j}}/\varepsilon^{n_{q_j}}) & 0_{n_{q_j}\times(k-j)} \end{bmatrix} + L_{q_jq}.$$

As an preliminary step toward our main result in Section 5, we present a lemma regarding the error dynamics of the high-gain observer for the extended system.

Lemma 2. *Assuming* $\hat{\theta} \in \Theta$, *there exists* $0 < \varepsilon^* \le 1$ *such that for all* $0 < \varepsilon \le \varepsilon^*$, *the error dynamics* (4) *is input-to-state stable (ISS) with respect to* $\tilde{\theta}$.

Proof. See Appendix A. □

4. Parameter Estimator [16]

In this section we recall from our previous work [16] the form of the parameter estimator and the stability requirements placed upon it. These requirements cannot be met for all parameterizations; indeed, they represent a significant restriction on the class of systems that be handled by our method. To demonstrate that the requirements can nevertheless be met for a large class of physically relevant systems, a series of propositions applying to different nonlinearities are presented by Grip et al. [16]. These propositions are accompanied by several examples, including a detailed physical example regarding downhole pressure estimation during oil well drilling. Due to space constraints, the propositions and examples are not repeated here; however, a simulation example is given in Section 7.

The parameter estimator is designed as though x and ϕ were known, with the goal of dynamically inverting the nonlinear equation $\phi = g(\nu, x, \theta)$ with respect to θ. In reality, x and ϕ are not known, and the parameter estimator is therefore implemented using the estimates $\hat{x}$ and $\hat{\phi}$ instead. In particular, the parameter estimator takes the form of an update law

$$\dot{\hat{\theta}} = u_\theta(\nu, \hat{x}, \hat{\phi}, \hat{\theta}). \tag{8}$$

The corresponding error dynamics is given by

$$\dot{\tilde{\theta}} = -u_\theta(\nu, \hat{x}, \hat{\phi}, \theta - \tilde{\theta}) \tag{9}$$

In the hypothetical case that $\hat{x} = x$ and $\hat{\phi} = \phi$, the parameter estimator should provide an unbiased asymptotic estimate of θ. This requirement is formally stated by the following assumption.

Assumption 4. *There exist a differentiable function* $V_\mathrm{u}\colon \mathbb{R}_{\geq 0} \times (\Theta - \Theta) \to \mathbb{R}_{\geq 0}$ *and positive constants* $a_1, \dots, a_4$ *such that for all* $(t, \tilde{\theta}) \in \mathbb{R}_{\geq 0} \times (\Theta - \Theta)$*,*

$$a_1\|\tilde{\theta}\|^2 \leq V_\mathrm{u}(t, \tilde{\theta}) \leq a_2\|\tilde{\theta}\|^2, \tag{10}$$

$$\frac{\partial V_\mathrm{u}}{\partial t}(t, \tilde{\theta}) - \frac{\partial V_\mathrm{u}}{\partial \tilde{\theta}}(t, \tilde{\theta}) u_\theta(\nu(t), x(t), \phi(t), \theta - \tilde{\theta}) \leq -a_3\|\tilde{\theta}\|^2, \tag{11}$$

$$\left\| \frac{\partial V_\mathrm{u}}{\partial \tilde{\theta}}(t, \tilde{\theta}) \right\| \leq a_4\|\tilde{\theta}\|. \tag{12}$$

Furthermore, the update law (8) *ensures that if* $\hat{\theta}(0) \in \Theta$*, then for all* $t \geq 0$*,* $\hat{\theta}(t) \in \Theta$*.*

5. Stability of Interconnected System

When analyzing the parameter estimator together with the high-gain observer for the extended system, we need one additional assumption.

Assumption 5. *The parameter update law* $u_\theta(\nu, \hat{x}, \hat{\phi}, \hat{\theta})$ *is Lipschitz continuous in* $(\hat{x}, \hat{\phi})$*, uniformly in* $(\nu, \hat{\theta})$*, on* $V \times \mathbb{R}^n \times \mathbb{R}^k \times \Theta$*.*

Remark 3. *The Lipschitz condition in Assumption 5 is global with respect to $\hat{x}$ and $\hat{\phi}$. Such a condition may fail to hold in many cases. However, if $u_\theta(\nu, \hat{x}, \hat{\phi}, \hat{\theta})$ is locally Lipschitz continuous in $(\hat{x}, \hat{\phi})$, uniformly in $(\nu, \hat{\theta})$, on $V \times \mathbb{R}^n \times \mathbb{R}^k \times \Theta$, then the update law can be modified to satisfy Assumption 5 by introducing a saturation on $\hat{x}$ and $\hat{\phi}$ outside the compact sets X and Φ. We also remark that when checking Assumption 5, the projection in the update law may be disregarded [16, App. A].*

Theorem 1. *There exists $0 < \varepsilon^* \leq 1$ such that for all $0 < \varepsilon \leq \varepsilon^*$, the origin of the error dynamics* (4), (9) *is exponentially stable, with all initial conditions such that $\hat{\theta}(0) \in \Theta$ contained in the region of attraction.*

Proof. See Appendix A. □

Remark 4. *The high-gain observer for the extended system estimates both the state x and the perturbation ϕ. In many cases, less noisy state estimates can be obtained by implementing a second observer that does not include a perturbation estimate, as $\dot{\hat{x}} = A\hat{x} + Bu + Eg(\nu, \hat{x}, \hat{\theta}) + K_x(\varepsilon)(y - C\hat{x})$, with $\hat{\theta}$ obtained from the first observer. The second observer can be designed by applying the same high-gain methodology to the error dynamics $\dot{\tilde{x}} = A\tilde{x} + E\tilde{g} - K_x(\varepsilon)\tilde{y}$, where $\tilde{g} := g(\nu, x, \theta) - g(\nu, \hat{x}, \hat{\theta})$. In this case, we need to impose the same Lipschitz-like assumptions on g as we previously did on d.*

6. Design Considerations

The design methodology in Section 3 ensures stability of the overall estimator by making ε sufficiently small. A bound on ε can be explicitly computed from the data of the system; however, such a computation is rarely useful, as it is complicated and is likely to yield a conservative result. Instead, the observer is typically tuned by starting with $\varepsilon = 1$ and decreasing ε in small decrements until satisfactory performance is achieved.

Part of the design involves a structural decomposition of the linear part of the system into the SCB, which explicitly displays the system's zero structure and invertibility properties. The zero structure of a system can be dramatically altered by infinitesimally small changes to the system matrices, for example, if such a change alters the system's relative degree [27]. If a small change in the system matrices results in a change in the zero structure, it can lead to a poorly conditioned gain-selection problem that results in unnecessarily high gains. In such cases, it can be beneficial to design the gains by omitting the change to the system matrices that is the cause of the structural bifurcation. An example of such a situation is given in Section 7.

7. Simulation Example

We consider the example of a DC motor with friction modeled by the LuGre friction model. This example, as well as the parameter values in Table 1, is borrowed from Canudas de Wit and Lischinsky [28]. The model is described by $\dot{\Omega} = \omega$, $J\dot{\omega} = u - F$, where Ω is the measured angular position, ω is the angular velocity, u is the motor torque, F is the friction torque, and J is the motor and load inertia. The friction torque—which may be due to friction in the motor itself, as well as the characteristics of the load and transmissions such as gears, links and joints—is given by the dynamic LuGre friction model: $F = \sigma_0\eta + \sigma_1\dot{\eta} + \alpha_2\omega$, where the internal friction state η is given by $\dot{\eta} = \omega - \sigma_0\eta|\omega|/\zeta(\omega)$, with $\zeta(\omega) = \alpha_0 + \alpha_1 \exp(-(\omega/\omega_0)^2)$. We assume that the parameters in Table 1 are known, except for the uncertain parameter $\theta := \alpha_0$, which represents Coloumb friction. To indicate that ζ depends on the unknown parameter, we shall henceforth write $\zeta(\omega, \theta)$. We assume that θ is known to belong to the range $\Theta := [0.03\,\mathrm{Nm}, 1\,\mathrm{Nm}]$. Following the notation from previous sections, we write $x = \mathrm{col}(\Omega, \omega, \eta)$ and $y = \Omega$. Let us define the perturbation $\phi = g(v, x, \theta) := \sigma_0\eta|\omega|/\zeta(\omega, \theta)$. It is straightforward to confirm that the system with input ϕ and output Ω is left-invertible and minimum-phase.

A technical problem arises due to the presence of $|\omega|$ in the perturbation, which causes the Lipschitz-like condition on d in Assumption 3 to fail. In the observer we therefore approximate the absolute value function by $|\omega| \approx a(\omega) := \omega^2/(2\kappa) + \kappa/2$ on the interval $[-\kappa, \kappa]$, where κ is a small number that we choose as 0.1. This approximation is continuously differentiable.

7.1. Estimator Design

We start the design procedure by extending the system to include the perturbation as a state, which yields the extended system

$$\begin{bmatrix} \dot{\Omega} \\ \dot{\omega} \\ \dot{\eta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{J}(\alpha_2 + \sigma_1) & -\frac{1}{J}\sigma_0 & \frac{1}{J}\sigma_1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Omega \\ \omega \\ \eta \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d. \tag{13}$$

Constructing an observer of the form (3) and transforming the corresponding error dynamics to the SCB, we obtain

$$\begin{aligned} \dot{\chi}_a &= -\frac{\sigma_0}{\sigma_1}\chi_a - \frac{\sigma_0^2(-\sigma_1\alpha_2 + J\sigma_0)}{\sigma_1^4}\gamma - K_a(\varepsilon)\gamma, \\ \dot{\chi}_{q_1} &= \chi_{q_2} - K_{q_1}(\varepsilon)\gamma, \quad \dot{\chi}_{q_2} = \chi_{q_2} - K_{q_2}(\varepsilon)\gamma, \\ \dot{\chi}_{q_3} &= \frac{\sigma_0^2(-\sigma_1\alpha_2 + J\sigma_0)}{J\sigma_1^3}\chi_{q_1} + \frac{\sigma_0(\sigma_1\alpha_2 - J\sigma_0)}{J\sigma_1^2}\chi_{q_2} \end{aligned}$$

$$+\frac{J\sigma_0-\sigma_1\alpha_2-\sigma_1^2}{\sigma_1 J}\chi_{q_3}+\frac{\sigma_0}{J}\chi_a+\delta-K_{q_3}(\varepsilon)y,\quad y=\chi_{q_1}.$$

We may now proceed to design the gains according to Section 3.1. However, we quickly discover that unacceptably large gains are required to stabilize the system. This problem is due to poor conditioning caused by the small parameter σ_1 appearing in the denominator in the SCB system equations, even though it does not appear in any denominators in the original system equations. The constant σ_1 can be viewed as a small perturbation to the system matrices in (13), which fundamentally alters the zero structure of the system by reducing the relative degree (see [27]). As suggested in Section 6, it is in this case better to design the gains using the approximation $\sigma_1 = 0$, which yields a simplified design system. After transformation to the SCB, the resulting error dynamics is given by

$$\dot{\chi}_{q_1}=\chi_{q_2}-K_{q_1}(\varepsilon)y,\quad \dot{\chi}_{q_2}=\chi_{q_3}-K_{q_2}(\varepsilon)y,$$
$$\dot{\chi}_{q_3}=\chi_{q_4}-K_{q_3}(\varepsilon)y,\quad \dot{\chi}_{q_4}=-\frac{\sigma_0}{J}\chi_{q_3}-\frac{\alpha_2}{J}\chi_{q_4}+\delta-K_{q_4}(\varepsilon)y,\quad y=\chi_{q_1}.$$

We now design the gains according to Section 3.1, placing the poles of $H = A_q - \bar{K}_q C_q$ at $-1 \pm 0.2j$ and $-2 \pm 0.2j$.

To design the parameter estimator, we note that in the hypothetical case that x and ϕ are known, the equation $\phi = g(v, x, \theta)$ can be solved explicitly with respect to θ as $\theta = \sigma_0\eta|\omega|/\phi - \alpha_1\exp(-(\omega/\omega_0)^2)$, assuming that $\phi \neq 0$. We may therefore design our parameter estimator based on the proposition by Grip et al. [16, Prop. 2] that applies to cases where an explicit solution is available part of the time, but not the whole time. According to this proposition the parameter estimate is designed to be exponentially attracted to the explicit solution when the solution is available, and it is otherwise kept constant. This yields the parameter estimator $\dot{\hat{\theta}} = \mathrm{Proj}(l(v,\hat{x},\hat{\phi})\Gamma(\theta^*(v,\hat{x},\hat{\theta}) - \hat{\theta}))$, where $\mathrm{Proj}(\,\cdot\,)$ denotes the parameter projection and θ^* represents the algebraic solution $\theta^*(v,\hat{x},\hat{\phi}) := \sigma_0\hat{\eta}a(\hat{\omega})/\hat{\phi} - \alpha_1\exp(-(\hat{\omega}/\omega_0)^2)$ based on the estimates from the high-gain observer. The function $l(v,\hat{x},\hat{\phi})$ is used to turn estimation off when a solution is unavailable or poorly conditioned, and is defined as $l(v,\hat{x},\hat{\phi}) = 0$ when $|\hat{\phi}| \leq 0.9$, and $l(v,\hat{x},\hat{\phi}) = 1$ when $|\hat{\phi}| \geq 1$, with a linear transition between 0 and 1 for $0.9 < |\hat{\phi}| < 1$. We select the gain as $\Gamma = 1$. It is easily seen that for $\hat{x} = x$ and $\hat{\phi} = \phi$, we obtain the error dynamics $\dot{\tilde{\theta}} = -\mathrm{Proj}(l(v,x,\phi)\tilde{\theta})$. We may conclude that this approach works if $|\phi| \geq 1$ is guaranteed to occur some portion of the time (see [16, Prop. 2]).

For this example, the Lipschitz-type conditions in Assumptions 3 and 5 are not satisfied globally with respect to $\hat{x}$ and $\hat{\phi}$. This issue can be rectified by saturating $\hat{x}$ and $\hat{\phi}$ outside a domain of interest, as discussed in Remarks 1 and 3.

7.2. Simulation Results

We simulate the system with the output y corrupted by noise, as shown in Figure 2. Using $\varepsilon = 0.5$ gives stable estimates and results in the gains $K(\varepsilon) \approx [12, 52, 0, 52]^\mathsf{T}$. To improve the state estimates, we create a second observer according to Remark 4. In this case, we do not need the Lipschitz condition on d, and we can implement $g(\nu, \hat{x}, \hat{\theta})$ without using the continuously differentiable approximation of the absolute value. For the second observer we place the poles of the matrix H at $-1 \pm 0.2j$ and -2 and select $\varepsilon = 1$, which gives the gains $K(\varepsilon) \approx [4, 5, 0]^\mathsf{T}$. The resulting state and parameter estimation errors are plotted in Figure 3.

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A. Proofs

Lemma 1. The error dynamics (4) with $K(\varepsilon) = 0$ consists of a system described by the left-invertible, minimum-phase triple (C, A, E), augmented by adding an integrator at each input point. Because integrators are left-invertible, it follows from the definition of left-invertibility that the augmented system is also left-invertible. The invariant zeros of a left-invertible triple (C, A, E) are the values of z for which the Rosenbrock system matrix $\left[\begin{smallmatrix} zI-A & -E \\ -C & 0 \end{smallmatrix}\right]$ loses rank. It is easy to confirm that when the triple (C, A, E) is augmented with an integrator at each input point, the resulting Rosenbrock system matrix loses rank for the same values of z as before, and hence the invariant zeros (and the minimum-phase property) remain the same. □

Lemma 2. This proof is based on the theory of Saberi and Sannuti [21]. Define $\xi_a = \chi_a$, $\xi_b = \chi_b$, and $\xi_q = \operatorname{col}(\xi_{q_1}, \dots, \xi_{q_k})$, where for each $j \in 1, \dots, k$, $\xi_{q_j} := S_j \chi_{q_j}$, with $S_j = \varepsilon^{\bar{n}_q - n_{q_j}} \operatorname{diag}(1, \dots, \varepsilon^{n_{q_j}-1})$ and $\bar{n}_q := \max_{j \in 1,\dots,k} n_{q_j}$. We obtain the following equations:

$$\dot{\xi}_a = A_a \xi_a, \quad \dot{\xi}_b = (A_b - K_{bb} C_b)\xi_b, \tag{14a}$$

$$\varepsilon \dot{\xi}_{q_j} = H_j \xi_{q_j} + B_{q_j} \varepsilon^{\bar{n}_q} (D_{a_j} \xi_a + D_{b_j} \xi_b + \delta_j + D^{\varepsilon}_{q_j} \xi_q), \tag{14b}$$

where $D^{\varepsilon}_{q_j} = D_{q_j} \operatorname{diag}(S_1^{-1}, \dots, S_k^{-1})$. Let P_a, P_b, and P_{q_j}, $j = 1, \dots, k$, be the symmetric, positive-definite solutions of the Lyapunov equations $P_a A_a + A_a^\mathsf{T} P_a =$

$-I_{n_a}$, $P_b(A_b - K_{bb}C_b) + (A_b - K_{bb}C_b)^\top P_b = -I_{n_b}$, and $P_{q_j}H_j + H_j^\top P_{q_j} = -I_{n_{a_j}}$, respectively. Define $W = \xi_a^\top P_a \xi_a + \xi_b^\top P_b \xi_b + \varepsilon \sum_{j=1}^k \xi_{q_j}^\top P_{q_j} \xi_{q_j}$. We then have

$$\begin{aligned}\dot{W} \le -\|\xi_a\|^2 - \|\xi_b\|^2 - \sum_{j=1}^k \|\xi_{q_j}\|^2 + \varepsilon^{\bar{n}_q} \sum_{j=1}^k 2\|P_{q_j}\| \\ \times (\|D_{a_j}\|\|\xi_a\| + \|D_{b_j}\|\|\xi_b\| + \|\delta_j\| + \|D_{q_j}^\varepsilon\|\|\xi_q\|)\|\xi_{q_j}\|.\end{aligned}$$

From the Lipschitz-like condition on d from Assumption 3, we know that for each $j \in 1,\dots,k$, there exist constants β_{a_j}, β_{b_j}, β_{θ_j}, and β_{q_j} such that $\|\delta_j\| \le \beta_{a_j}\|\chi_a\| + \beta_{b_j}\|\chi_b\| + \beta_{\theta_j}\|\tilde{\theta}\| + \beta_{q_j}\|\chi_q\|$, which means that $\|\delta_j\| \le \beta_{a_j}\|\xi_a\| + \beta_{b_j}\|\xi_b\| + \beta_{\theta_j}\|\tilde{\theta}\| + \varepsilon^{-(\bar{n}_q-1)}\beta_{q_j}\|\xi_q\|$. We furthermore have $\|D_{q_j}^\varepsilon\| \le \varepsilon^{-(\bar{n}_q-1)}\|D_{q_j}\|$.

Let $\rho_a = \sum_{j=1}^k 2\|P_{q_j}\|(\|D_{a_j}\| + \beta_{a_j})$, $\rho_b = \sum_{j=1}^k 2\|P_{q_j}\|(\|D_{b_j}\| + \beta_{b_j})$, $\rho_q = \sum_{j=1}^k 2\|P_{q_j}\|(\|D_{q_j}\| + \beta_{q_j})$, and $\rho_\theta = \sum_{j=1}^k 2\|P_{q_j}\|\beta_{\theta_j}$. Then we may write

$$\begin{aligned}\dot{W} \le -\|\xi_a\|^2 - \|\xi_b\|^2 - (1 - \varepsilon\rho_q)\|\xi_q\|^2 \\ + \varepsilon^{\bar{n}_q}\rho_a\|\xi_a\|\|\xi_q\| + \varepsilon^{\bar{n}_q}\rho_b\|\xi_b\|\|\xi_q\| + \varepsilon^{\bar{n}_q}\rho_\theta\|\tilde{\theta}\|\|\xi_q\|.\end{aligned}$$

Note that ρ_q is multiplied by ε. Furthermore, note that the cross terms between $\|\xi_a\|$ and $\|\xi_q\|$, and between $\|\xi_b\|$ and $\|\xi_q\|$, are multiplied by $\varepsilon^{\bar{n}_q}$ relative to the stabilizing quadratic terms in $\|\xi_a\|^2$, $\|\xi_b\|^2$, and $\|\xi_q\|^2$. It is therefore straightforward to show that, by decreasing ε, the cross terms are dominated, and there exist positive constants c_a, c_b, and c_q such that $\dot{W} \le -c_a\|\xi_a\|^2 - c_b\|\xi_b\|^2 - c_q\|\xi_q\|^2 + \varepsilon^{\bar{n}_q}\rho_\theta\|\tilde{\theta}\|\|\xi_q\|$. This expression shows that $\dot{W}$ is negative outside a ball around the origin $(\xi_a, \xi_b, \xi_q) = 0$, the size of which is proportional to $\tilde{\theta}$. We therefore conclude that (14) is ISS with respect to $\tilde{\theta}$ [29, Th. 4.19]. Since (14) is obtained through a nonsingular transformation of (4), the same holds for (4). □

Theorem 1. Assumption 4 ensures that since $\hat{\theta}(0) \in \Theta$, we have $\hat{\theta}(t) \in \Theta$ for all $t \ge 0$. From Lemma 2, the error dynamics (4) is ISS with respect to $\tilde{\theta}$. Hence, the trajectory of (4), (9) remains in a compact subset of $\mathbb{R}^{n+k} \times (\Theta - \Theta)$ for all future time. Based on Assumption 5, we know, by following the same argument as for $\|\delta_j\|$ in the proof of Lemma 2, that there exist positive constants $\bar{\beta}_a$, $\bar{\beta}_b$, $\bar{\beta}_q$ such that $\|u_\theta(\nu, x, \phi, \hat{\theta}) - u_\theta(\nu, \hat{x}, \hat{\phi}, \hat{\theta})\| \le \bar{\beta}_a\|\xi_a\| + \bar{\beta}_b\|\xi_b\| + \varepsilon^{-(\bar{n}_q-1)}\bar{\beta}_q\|\xi_q\|$. Define $V := W + \varepsilon^{2\bar{n}_q-1}V_u$, where W is from the proof of Lemma 2. We then obtain

$$\begin{aligned}\dot{V} \le -c_a\|\xi_a\|^2 - c_b\|\xi_b\|^2 - c_q\|\xi_q\|^2 - a_3\varepsilon^{2\bar{n}_q-1}\|\tilde{\theta}\|^2 \\ + \varepsilon^{\bar{n}_q}[(\rho_\theta + a_4\bar{\beta}_q)\|\xi_q\| + a_4\bar{\beta}_a\|\xi_a\| + a_4\bar{\beta}_b\|\xi_b\|]\|\tilde{\theta}\|.\end{aligned}$$

To show that we may dominate all the cross terms in the expression above by decreasing ε, note that for arbitrary positive constants α_1, α_2, and α_3, the expression $-\alpha_1\zeta_1^2 - \alpha_2\varepsilon^{2\tilde{n}_q-1}\zeta_2^2 + \alpha_3\varepsilon^{\tilde{n}_q}\zeta_1\zeta_2$ can be made negative definite by selecting $\varepsilon < 4\alpha_1\alpha_2/\alpha_3^2$. Since we may split out expressions like these for each of the cross terms, by letting α_1 and α_2 be small fractions of the negative quadratic terms and letting α_3 be the constant in the cross term, it is clear that there exists ε^* and a constant $c > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, $\dot{V} \leq -c(\|\xi_a\|^2 + \|\xi_b\|^2 + \|\xi_q\|^2 + \|\tilde{\theta}\|^2)$. By invoking the comparison lemma [29, Lemma 3.4], we may therefore conclude that the origin of (14), (9) is exponentially stable with all initial conditions such that $\hat{\theta}(0) \in \Theta$ contained in the region of attraction. Since (14) is obtained through a nonsingular transformation of (4), the same holds for (4), (9). □

References

[1] M. Krstić, I. Kanellakopoulos, P. V. Kokotović, Nonlinear and Adaptive Control Design, Wiley, New York, 1995.

[2] R. Marino, P. Tomei, Nonlinear Control Design: Geometric, Adaptive and Robust, Prentice-Hall, Upper Saddle River, NJ, 1995.

[3] V. Fomin, A. Fradkov, V. Yakubovich (Eds.), Adaptive Control of Dynamical Systems, Nauka, Moskow, 1981.

[4] R. Ortega, Some remarks on adaptive neuro-fuzzy systems, Int. J. Adapt. Contr. Signal Process. 10 (1) (1996) 79–83.

[5] A. M. Annaswamy, F. P. Skantze, A.-P. Loh, Adaptive control of continuous time systems with convex/concave parametrization, Automatica 34 (1) (1998) 33–49.

[6] M. S. Netto, A. M. Annaswamy, R. Ortega, P. Moya, Adaptive control of a class of non-linearly parameterized systems using convexification, Int. J. Contr. 74 (14) (2000) 1312–1321.

[7] J. D. Bošković, Adaptive control of a class of nonlinearly parameterized plants, IEEE Trans. Automat. Contr. 43 (7) (1998) 930–934.

[8] T. Zhang, S. S. Ge, C. C. Hang, T. Y. Chai, Adaptive control of first-order systems with nonlinear parameterization, IEEE Trans. Automat. Contr. 45 (8) (2000) 1512–1516.

[9] Z. Qu, Adaptive and robust controls of uncertain systems with nonlinear parameterization, IEEE Trans. Automat. Contr. 48 (10) (2003) 1817–1823.

[10] I. Y. Tyukin, D. V. Prokhorov, C. van Leeuwen, Adaptation and parameter estimation in systems with unstable target dynamics and nonlinear parametrization, IEEE Trans. Automat. Contr. 52 (9) (2007) 1543–1559.

[11] C. Cao, A. Annaswamy, A hierarchical discretized-parameter polynomial adaptive estimator for non-linearly parameterized systems, Int. J. Contr. 79 (8) (2006) 831–844.

[12] W. Lin, C. Qian, Adaptive control of nonlinearly parameterized systems: The smooth feedback case, IEEE Trans. Automat. Contr. 47 (8) (2002) 1249–1266.

[13] X. Liu, R. Ortega, H. Su, J. Chu, Immersion and invariance adaptive control of nonlinearly parameterized nonlinear systems, in: Proc. American Contr. Conf., St. Louis, MO, 2009, pp. 1268–1273.

[14] A. Astolfi, R. Ortega, Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems, IEEE Trans. Automat. Contr. 48 (4) (2003) 590–606.

[15] X. Liu, R. Ortega, H. Su, J. Chu, On adaptive control of nonlinearly parameterized nonlinear systems: Towards a constructive procedure, Syst. Contr. Lett. 60 (1) (2011) 36–43.

[16] H. F. Grip, T. A. Johansen, L. Imsland, G.-O. Kaasa, Parameter estimation and compensation in systems with nonlinearly parameterized perturbations, Automatica 46 (1) (2010) 19–28.

[17] H. F. Grip, A. Saberi, T. A. Johansen, State and parameter estimation for linear systems with nonlinearly parameterized perturbations, in: Proc. IEEE Conf. Dec. Contr., Shanghai, China, 2009, pp. 8218–8225.

[18] L. B. Freidovich, H. K. Khalil, Lyapunov-based switching control of nonlinear systems using high-gain observers, Automatica 43 (1) (2007) 150–157.

[19] L. B. Freidovich, H. K. Khalil, Performance recovery of feedback-linearization-based designs, IEEE Trans. Automat. Contr. 53 (10) (2008) 2324–2334.

[20] A. Chakrabortty, M. Arcak, Time-scale separation redesigns for stabilization and performance recovery of uncertain nonlinear systems, Automatica 45 (1) (2009) 34–44.

[21] A. Saberi, P. Sannuti, Observer design for loop transfer recovery and for uncertain dynamical systems, IEEE Trans. Automat. Contr. 35 (8) (1990) 878–897.

[22] J. P. Gauthier, H. Hammouri, S. Othman, A simple observer for nonlinear systems: Applications to bioreactors, IEEE Trans. Automat. Contr. 37 (6) (1992) 875–880.

[23] X. Fan, M. Arcak, Observer design for systems with multivariable monotone nonlinearities, Syst. Contr. Lett. 50 (4) (2003) 319–330.

[24] R. Marino, P. Tomei, Global adaptive output-feedback control of nonlinear systems, part II: Nonlinear parameterization, IEEE Trans. Automat. Contr. 38 (1) (1993) 33–48.

[25] Z. Ding, Adaptive control of triangular systems with nonlinear parameterization, IEEE Trans. Automat. Contr. 46 (12) (2001) 1963–1968.

[26] P. Sannuti, A. Saberi, Special coordinate basis for multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling, Int. J. Contr. 45 (5) (1987) 1655–1704.

[27] S. Sastry, J. Hauser, P. V. Kokotović, Zero dynamics of regularly perturbed systems may be singularly perturbed, Syst. Contr. Lett. 13 (4) (1989) 299–314.

[28] C. Canudas de Wit, P. Lischinsky, Adaptive friction compensation with partially known dynamic friction model, Int. J. Adapt. Contr. Signal Process. 11 (1) (1997) 65–80.

[29] H. K. Khalil, Nonlinear Systems, 3rd Edition, Prentice-Hall, Upper Saddle River, NJ, 2002.