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On nonlinear unknown input observers – applied to lateral vehicle velocity estimation on banked roads

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Unknown input observers (UIOs) are observers that have stable error dynamics that are independent of unknown inputs. This paper studies such observers for nonlinear systems, and shows that the error dynamics for a nonlinear UIO has the same structure as the error dynamics of a nonlinear observer without unknown inputs. This result is first used to provide synthesis inequalities for UIOs for a class of nonlinear systems, and secondly, to inspire the design of an observer for estimation of vehicle lateral velocity on banked roads.

1 Introduction

Often, when estimating the state of state space models $\dot{x} = f(x)$ based on some measurements $y = g(x)$ that depend on the states, one is faced with the presence of unknown inputs or disturbances in the differential equations, $\dot{x} = f(x, w)$. A standard method in this case is to assume that some model of the parameter time variation is known (typically constant ($\dot{w} = 0$), Markov models or Brownian motion in a stochastic setting) and extend the state space model with this input model. Examples are the augmented extended Kalman filter (e.g. Gelb (1974)), or nonlinear observers with bias estimation (e.g. Vik and Fossen (2001)).

In this paper, another approach will be pursued, which does not assume known parameter dynamics, but rather relies on some geometric property of the system and output mapping. This property will allow us to specify observers that have error dynamics which are completely independent of the unknown input, denoted *Unknown Input Observers* (UIOs, see definition below). In the linear case, such observers are well known (e.g. Darouach et al. (1994), Chen et al. (1996)). For nonlinear systems, there exist some results on UIOs, mostly in connection with fault detection and isolation, see, e.g., Seliger and Frank (1991). Of more recent results, we mention Rocha-Cozatl et al. (2004) who give conditions for design of UIOs based on dissipativity, which for a special system class translate into LMIs (Rocha-Cozatl et al. 2005).

In this paper, we will give a simple extension of the results in the linear case to nonlinear systems. This extension is not reported in the literature, to the best of the authors' knowledge, probably because it does not provide constructive design guidelines. In a similar manner as for linear systems in Darouach et al. (1994), which showed that finding a UIO for linear systems can be formulated as the problem of finding a (Luenberger) observer for a linear system, we show that (for a certain, fairly general class of nonlinear systems) proving that a nonlinear UIO is stable is as hard as (but not structurally harder than) proving that a nonlinear observer is stable in general.

Thereafter, using this result, we provide a synthesis procedure for nonlinear UIOs based on the observer design for a class of nonlinear systems described in Arcak and Kokotović (2001).

Next, we show how the result inspire the re-design an observer for vehicle lateral velocity under the influence of road bank angle. This problem has to some extent been studied earlier in the context of vehicle dynamics control systems, for example using an augmented EKF (Suissa et al. 1994), using linear observers augmented with road bank angle estimation (Fukada 1999), estimating road bank angle separately from lateral velocity (Tseng 2001, Ungoren et al. 2004), or using additional measurements such as DGPS (Hahn

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et al. 2004). Linear UIO methods were used in Liu and Peng (2002). Our approach extends the results in Imsland et al. (2006) to banked roads, and differs from earlier approaches since we use a nonlinear friction model and analyze stability of the observer error dynamics. In addition to directly applying the nonlinear UIO theory, we also propose a method that estimates the unknown input. This observer is tested on real vehicle data.

The following definition of Unknown Input Observers is adapted from Moreno (2000) (with abuse of notation and some standard technicalities left out for brevity):

Definition 1.1 Given the (possibly) nonlinear system $\dot{x} = f(x, u, w)$, $y = h(x, u)$, with x the state, u known input, w unknown input and y measured output. The system (the *observer*)

$$\dot{z} = \phi(z, u, y),$$
$$\hat{x} = \xi(z, u, y),$$

is an unknown input observer (for the above system) if

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0,$$

uniformly in (that is, the rate of convergence is independent of) t_0 and $w(\cdot)$ (and initial conditions).

One technicality required is that solutions to the nonlinear system to be observed must exist for all future time. This is implicitly assumed (including the necessary smoothness-assumptions on f and h) throughout the paper.

2 Linear systems

This section reviews some results on UIOs for linear systems.

2.1 *UIOs for linear systems*

Consider the LTI system

$$\dot{x} = Ax + Bu + Dw, \quad y = Cx, \tag{1}$$

where x, u and w are the state, the known inputs and the unknown inputs, respectively, while y is the (measured) output. It is generally acknowledged (e.g. Darouach et al. (1994), Chen et al. (1996)) that full order UIOs for linear systems take the (Luenberger observer-like) form

$$\dot{z} = Nz + Ly + Gu, \quad \hat{x} = z - Ey. \tag{2}$$

The matrices N, L, G and E must be specified such that the dynamics for the observer error $\tilde{x} = x - \hat{x}$ can be written $\dot{\tilde{x}} = N\tilde{x}$, where N is Hurwitz. Then, the error system is (globally) exponentially stable and independent of the unknown input w, and hence the observer is a UIO according to Definition 1.1.

By constructive design, the following matrix equalities give a UIO[1]:

$$(I + EC)D = 0, \tag{3a}$$
$$G = (I + EC)B, \tag{3b}$$
$$N = (I + EC)A - L_1C, \tag{3c}$$
$$L = -NE + L_1. \tag{3d}$$

The existence conditions for solutions (for N (stable), L, G and E) are that $\operatorname{rank} CD = \operatorname{rank} D = \dim y$, and that the pair $((I + EC)A, C)$ is detectable (Darouach et al. 1994, Chen et al. 1996). The crucial identity to solve is (3a), which under the existence conditions is solved (for E) as

$$E = -D(CD)^+ + Y(I - CD(CD)^+), \tag{4}$$

where $(CD)^+$ is the left inverse of CD, $(CD)^+ = [(CD)^\mathsf{T} CD]^{-1}(CD)^\mathsf{T}$, and Y is an arbitrary matrix of appropriate dimensions. It is this operator that "decouples" the unknown input.

It is interesting to write the observer in terms of a differential equation for $\hat{x}$:

$$\begin{aligned}\dot{\hat{x}} &= \dot{z} - E\dot{y} \\ &= N(\hat{x} + Ey) + LCx + Gu - EC(Ax + Bu + Dw) \\ &= (I + EC)A\hat{x} - L_1C\hat{x} + NEy - NECx + L_1Cx + (I + EC)Bu - EC(Ax + Bu + Dw) \\ &= A\hat{x} + Bu + L_1(Cx - C\hat{x}) + EC(A\hat{x} + Bu) - EC(Ax + Bu + Dw).\end{aligned}$$

Since $(I + EC)D = 0$, the term Dw can be moved out from the last paranthesis, thus we can write this as

$$\dot{\hat{x}} = A\hat{x} + Bu + Dw + L_1(y - C\hat{x}) - ECA(x - \hat{x}).$$

This is essential, since the term Dw then disappears from the error dynamics,

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu + Dw - A\hat{x} - Bu - Dw - L_1(y - C\hat{x}) + ECA(x - \hat{x}) \\ &= A\tilde{x} - L_1C\tilde{x} + ECA\tilde{x} \\ &= N\tilde{x}.\end{aligned}$$

For stability, L_1 (and Y) must be chosen such that N is Hurwitz, which is always possible under the detectability assumption mentioned above. If we strengthen the assumption to observability of $((I + EC)A, C)$, we see that the eigenvalues of N can be placed arbitrarily. A procedure for constructing linear UIOs can be summarized as a) calculate E using (4), b) find L_1 such that $N = (I + EC)A - L_1C$ is Hurwitz, and c) calculate G (3b) and L (3d).

If we consider the case with $\dim y = \dim w = 1$, existence of an UIO requires relative degree (w.r.t. w) equal to one, since we must have $CD \neq 0$ for (3a) to be solvable. In higher dimensions we see from (4) that we must have CD left invertible, which for $\dim w = 1$ means that at least one measurement must depend directly on one of the states that w affects.

Finally, we stress again that the property (3a) factors out Dw from the error dynamics, since

$$E\dot{y} = EC\dot{x} = EC(Ax + Bu) + ECDw = EC(Ax + Bu) - Dw.$$

[1]Eliminating L_1, the two last equations can be written as $N = (I + EC)A - LC - NEC$, which gives the same conditions as Darouach et al. (1994).

2.2 *Disturbance observers for linear systems*

The above observer can be expanded to also provide an estimate of the disturbance $d = Dw$. Consider the estimate (very similar to what is proposed in Liu and Peng (2002)),

$$\hat{d} = L_1 y - E\dot{y} - (L_1 C - ECA)\hat{x} + ECBu.$$

Then,

$$\begin{aligned} Dw - \hat{d} &= Dw - L_1 y + EC(Ax + Bu + Dw) + (L_1 C - ECA)\hat{x} - ECBu \\ &= (I + EC)Dw - (L_1 C - ECA)\tilde{x} \\ &= -(L_1 C - ECA)\tilde{x}. \end{aligned}$$

Since $\tilde{x} \to 0$, we conclude that $\hat{d} \to Dw$. Furthermore, if D is full row rank (and $\dim d \leq \dim x$), w can be found from d.

It is noteworthy that the convergence of the estimate $\hat{d}$ to Dw is completely independent of w. A considerable practical disadvantage of this approach is that the estimate depends on the differentiated output, $\dot{y}$.

3 UIO for nonlinear systems

In this section, we use insights from the previous section to specify conditions for nonlinear UIOs.

First, note that if only nonlinearities that depend on known signals (inputs and outputs, i.e. of the form $\gamma(y, u)$) appear in the system dynamics, the same procedure as for linear UIOs gives a nonlinear UIO, as such nonlinearities can be canceled in the error dynamics. This section treats general nonlinearities that depend on unmeasured states, but for illustration, we have included a nonlinearity of the form $\gamma(y, u)$ in the design procedure in Section 4.

3.1 *Linear measurement equation*

Consider the nonlinear system

$$\dot{x} = f(x, u) + Dw, \quad y = Cx. \tag{5}$$

A nonlinear UIO for this system with structure[1] as in the LTI case, is

$$\dot{z} = (I + EC)f(z - Ey, u) + L_1(y - C(z - Ey)), \quad \hat{x} = z - Ey. \tag{6}$$

Assuming $(I + EC)D = 0$, it is easy to see that we can write

$$\dot{\hat{x}} = f(\hat{x}, u) + Dw + L_1(y - C\hat{x}) - EC\left(f(x, u) - f(\hat{x}, u)\right).$$

This gives the following observer error system:

$$\dot{\tilde{x}} = (I + EC)(f(x, u) - f(\hat{x}, u)) - L_1(y - C\hat{x}),$$

which is independent of w.

From the above, we conclude the following theorem:

[1] Replacing $f(x, u)$ with $Ax + Bu$ and using (3), we get (2).

Theorem 3.1 *Assume that* $\operatorname{rank} CD = \operatorname{rank} D = \dim y$ *for (5), and let the constant matrix* E *be given by (4). Then the system (6) is a nonlinear UIO for (5) if the constant matrices* Y *and* L_1 *are chosen such that* $\tilde{x} = 0$ *is a uniformly globally asymptotically stable equilibrium of the system*

$$\dot{\tilde{x}} = (I + EC)(f(x,u) - f(x - \tilde{x}, u)) - L_1 C \tilde{x}.$$

Thus, as long as the rank-assumption holds, designing L_1 to get a UGAS error system is in principle as hard as (but not harder than) designing a UGAS observer with linear injection term in general. The corresponding conclusion was made for linear UIOs in Darouach et al. (1994).

3.2 *Nonlinear measurement equation*

Assume now that the system is given by

$$\dot{x} = f(x,u) + Dw, \quad y = h(x). \tag{7}$$

Since the measurement equation is not linear, we cannot apply (3a) directly, as we did in the previous section. Replacing the linear mapping E with a nonlinear mapping $e(y)$, we propose the following structure for the nonlinear UIO:

$$\dot{z} = g(z,y,u), \quad \hat{x} = z - e(y).$$

where the nonlinear functions $g(z,y,u)$ and $e(y)$ are to be found. We do not consider measurement equations that depends on the input, $h(x,u)$ (or a mapping $e(y,u)$ instead of $e(y)$), since this will result in nonlinear UIOs that depend on the time derivative of the inputs.

Proceeding as in the previous section, it is easy to show that if we can find $e(y)$ such that, similar to (3a),

$$\left(I + \frac{\partial e}{\partial y}\frac{\partial h}{\partial x}\right) D = 0, \quad \forall x, \tag{8}$$

then we can design a UIO. The natural choice for g is then

$$g(z,y,u) = \left(I + \frac{\partial e}{\partial y}\frac{\partial h}{\partial x}\right) f(z - e(y), u) + L_1(y - h(z - e(y))).$$

We see that such an observer is only possible if $\frac{\partial e}{\partial y}\frac{\partial h}{\partial x}$ is independent of x, which in general is hard to obtain. However, in some cases, the structure of $e(y)$ can be chosen such that this condition holds, for example if one has a mixture of linear and nonlinear measurements. Then the linear measurements can be used to decouple the unknown input, while the nonlinear (and linear) measurements can be used to stabilize the error dynamics.

As an example, consider the system

$$\dot{x} = f(x,u) + Dw, \quad y = \left(y_1^\mathsf{T}, y_2^\mathsf{T}\right)^\mathsf{T} = \left((Cx)^\mathsf{T}, h(x)^\mathsf{T}\right)^\mathsf{T}.$$

For this system, we propose the UIO

$$\dot{z} = (I + EC) f(z - Ey_1, u) + L_1(y_1 - C(z - Ey_1)) + L_2(y_2 - h(z - Ey_1)), \quad \hat{x} = z - Ey_1. \tag{9}$$

Proceeding as in Section 3.1, if we can find E such that $(I + EC)D = 0$, then the observer error dynamics can be written

$$\dot{\tilde{x}} = (I + EC)((f(x,u) - f(x - \tilde{x}, u)) - L_1 C\tilde{x} - L_2(h(x) - h(x - \tilde{x})),$$

which is independent of the unknown input w. Notably, we can use all measurements (both L_1 and L_2) to stabilize the error dynamics, while only the linear measurement y_1 is used to make the observer error independent of w.

3.3 *Toward constructive design*

As mentioned above, the previous sections give conditions for an observer to be a nonlinear UIO, and do not give constructive guidelines. However, there exist in the literature some design procedures for nonlinear observers, which by Theorem 3.1 can also be used for design of nonlinear UIOs. Most notably, perhaps, are the results of Arcak and Kokotović (2001), Fan and Arcak (2003), which we combine with Theorem 3.1 in the next section.

Observer design procedures for a certain class of output nonlinearities can be found in Fan and Arcak (2003). These can be combined with the results in Section 3.2, but as the combination is similar to the next section, we do not go into detail on this.

In Section 5, we will look at how we can apply the insights gained in this section to the problem of estimating lateral velocity of cars on banked roads.

4 Nonlinear UIOs based on circle criteria observers

Consider now nonlinear systems on the form

$$\dot{x} = Ax + G\rho(Hx) + \gamma(y, u) + Dw, \quad y = Cx, \tag{10}$$

where the first nonlinearity $\rho : \mathbb{R}^p \to \mathbb{R}^p$ satisfies a monotonicity property to be specified.

We will use the results in Arcak and Kokotović (2001), Fan and Arcak (2003) to define synthesis inequalities which upon feasibility gives observer injection gains which stabilize the observer error dynamics of Theorem 3.1.

4.1 *Observer and synthesis inequality*

We propose the following observer for (10):

$$\begin{aligned}\dot{z} = (I + EC)A(z - Ey) + (I + EC)G\rho\left(H(z - Ey) + K(y - C(z - Ey))\right) \\ + L_1(y - C(z - Ey)) + (I + EC)\gamma(y, u), \quad \hat{x} = z - Ey, \end{aligned} \tag{11}$$

where E is given by (4) such that $(I + EC)D = 0$. A slight change is made compared to the previous section, we use an additional injection gain K, following the approach of Arcak and Kokotović (2001). Differentiating the estimated state gives

$$\begin{aligned}\dot{\hat{x}} = A\hat{x} + G\rho(H\hat{x} + K(y - C\hat{x})) + \gamma(y, u) + Dw \\ - ECA(x - \hat{x}) - ECG\left(\rho(Hx) - \rho(H\hat{x} + K(y - C\hat{x}))\right) + L_1(y - C\hat{x}).\end{aligned}$$

This gives the following observer error system, independent of w:

$$\dot{\tilde{x}} = (I + EC)A\tilde{x} + (I + EC)G\left(\rho(Hx) - \rho(H\hat{x} + KC\tilde{x})\right) - L_1C\tilde{x}. \tag{12}$$

If we define

$$\phi(v, \zeta) = \rho(v) - \rho(v - \zeta), \quad v := Hx,$$

the error system (12) can be rewritten as

$$\dot{\tilde{x}} = ((I + EC)A - L_1 C)\,\tilde{x} + (I + EC)G\phi(v, \zeta), \quad \zeta = (H - KC)\tilde{x}.$$

Assumption 4.1 The function $\rho(v)$ satisfies the monotonicity property

$$\frac{\partial \rho}{\partial v} + \left(\frac{\partial \rho}{\partial v}\right)^{\mathsf{T}} \geq 0, \quad \forall v \in \mathbb{R}^p.$$

This assumption implies that $\phi(v, \zeta)$ satisfies the sector property $\zeta^{\mathsf{T}}\phi(v, \zeta) \geq 0$, and this allows to conclude asymptotic stability if the linear system with input $-\phi(v, \zeta)$ and output ζ is strictly positive real (SPR), which is implied by the existence of $P > 0$ and a constant $\kappa > 0$ such that

$$\begin{pmatrix} ((I + EC)A - L_1 C)^{\mathsf{T}} P + P((I + EC)A - L_1 C) + \kappa I & P(I + EC)G + (H - KC)^{\mathsf{T}} \\ G^{\mathsf{T}}(I + EC)^{\mathsf{T}} P + (H - KC) & 0 \end{pmatrix} \leq 0. \tag{13}$$

This inequality is a linear matrix inequality (LMI) in P, PL_1, K, and κ. Since $P > 0$, L_1 can be obtained from PL_1.

THEOREM 4.2 *If observer injection gains L_1 and K, matrices $P > 0$ and $\kappa > 0$ exist such that (13) is feasible, then the observer (11) is a UIO, whose error dynamics are UGES.*

Proof [Outline] The proof follows by checking that the SPR condition guarantees that the derivative of the Lyapunov function $V(\tilde{x}) = \tilde{x}^{\mathsf{T}} P\tilde{x}$ is negative definite along the trajectories of (12), implying that the origin of the observer error dynamics, which by design are independent of the unknown input w, are uniformly globally exponentially stable (and thus uniformly convergent). □

Remark 1 In Fan and Arcak (2003), the observer design in Arcak and Kokotović (2001) is extended in two directions. The same can be done for the design in this section. First, if it is possible to find a structured multiplier matrix Λ such that $(\Lambda\zeta)^{\mathsf{T}}\phi(v, \zeta) \geq 0$ for all $\Lambda > 0$ with this particular structure, then the structured matrix Λ can be used to generalize (13). This applies if the multivariable nonlinearity $\rho(v)$ has a decoupled structure.

Moreover, synthesis inequalities for the system class (10) with additional (monotone) output nonlinearities are also specified. This design can be combined with the approach in Section 3.2 to give synthesis inequalities for existence of a UIO. As the procedure is similar to that above, the details are not given here.

Remark 2 Rocha-Cozatl et al. (2005) develops a synthesis SDP for nonlinear UIOs for essentially the same class of system that is considered in this section. Although both the approach herein and the approach of Rocha-Cozatl et al. (2005) is based on dissipativity and result in LMIs, the synthesis inequalities have different structure and it is not straightforward, for a given problem, to say whether one approach is less conservative than the other. However, the approach in Rocha-Cozatl et al. (2005) is more general in the sense that several types of dissipativity can be exploited, while the approach herein is simpler; it has fewer optimization variables, and is a more direct extension of linear UIO design, since it uses the property (3a).

4.2 *Example using circle criteria design*

Consider the mass-damper system taken from Rocha-Cozatl et al. (2005),

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -\frac{1}{m_1} \\ 0 \\ 0 \end{pmatrix} \rho(x_2) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{pmatrix} w$$

where we measure x_1 and x_4, and the nonlinearity ρ is due to a nonlinear damping device, $\rho(v) = c_{\text{nl}}\,\text{sign}(v)\ln(1+|v|)$. This system is of the class (10), and since $d\rho/dv > 0$ always, Assumption 4.1 holds. The aim of this example is to illustrate the UIO design in Theorem 4.2, by designing a UIO for estimating the unmeasured state variables x_2 and x_3. Parameter values can be found in Rocha-Cozatl et al. (2005).

Using LMI solvers in Yalmip (Löfberg 2004), we find that the output injections

$$L_1 = \begin{pmatrix} -0.3501 & 0.7234 \\ -4.5838 & -0.2351 \\ 0.2936 & 1.4022 \\ -0.8632 & 0.5000 \end{pmatrix}, \quad K = \begin{pmatrix} -0.8780 & 0 \end{pmatrix},$$

renders the LMI feasible, implying exponential stability of the error dynamics for the UIO (11).

Simulating the system with the nonlinear UIO, letting $u(t) = 5$ and

$$w(t) = \begin{cases} 0, & t < 10, \\ 4\sin t + 2, & t \geq 10, \end{cases}$$

we can confirm that the UIO error dynamics is unaffected by the unknown input (see Figure 1). Compared to the UIO in Rocha-Cozatl et al. (2005), the structure of the resulting observer is the same, but the SDP used for synthesis here has slightly fewer variables (the variable $\mathbb{S}$ in Rocha-Cozatl et al. (2005) does not have its counterpart in (13)).

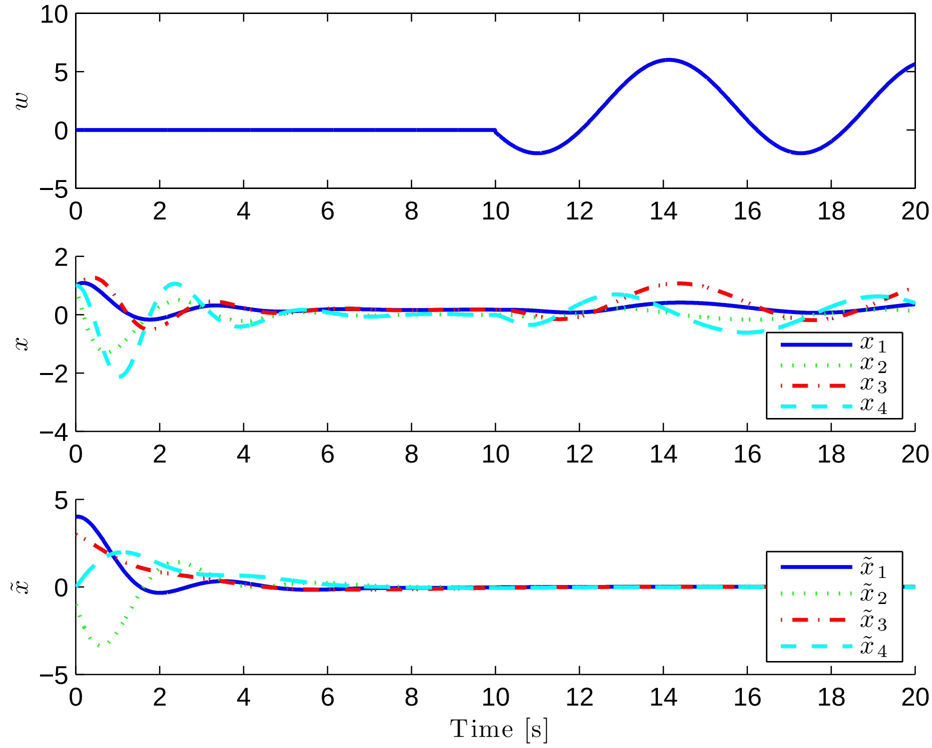

Figure 1. Unknown input, states, and observer error

5 Estimation of vehicle lateral velocity on banked roads

Many systems for automotive active safety/vehicle dynamics control depend on information about vehicle lateral velocity, v_y. It is generally acknowledged that to measure v_y directly is either too expensive or too unreliable; thus some kind of estimation algorithm, or observer, must be used. The standard measurement suite used for obtaining an estimate of v_y consists of wheel speed measurements ω_i (which can be used to calculate an estimate of longitudinal velocity v_x), steering wheel angle δ (used to calculate individual wheel angles δ_i), yaw rate r and lateral acceleration a_y. All these measurements can be considered standard in modern cars with yaw stabilization (anti-skid) systems, such as the Bosch ESP system (van Zanten 2000). The acceleration and yaw rate measurement are typically obtained from MEMS-based sensors which we assume is located in the center of gravity (or corrected if this is not the case), and to have bias and slow drift components removed by appropriate filtering. Observers for v_y can be found in the literature, e.g. (Kiencke and Nielsen 2000, Imsland et al. 2006), and are also implemented in production cars (van Zanten 2000). However, except for the Extended Kalman Filter (EKF) (Suissa et al. 1994), the authors are not aware of any approaches reported in the literature that use nonlinear techniques (models) for estimating lateral velocity taking banked roads explicitly into account. The main advantage of the approach herein compared to using EKF-based designs is lower computational complexity, since there is no Riccati equation to be solved online.

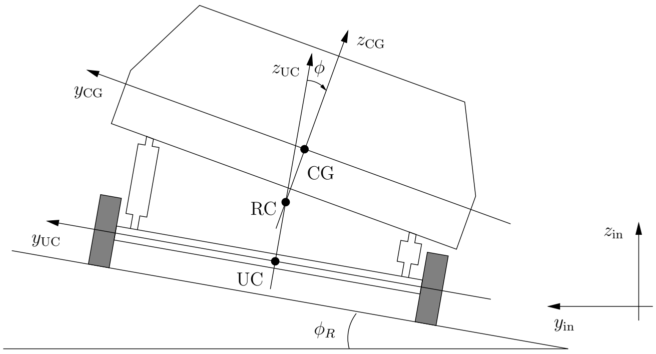

Figure 2. Vehicle seen from behind, illustrating the road bank angle and the different coordinate systems involved. The abbrieviations CG, RC and UC stands for center of gravity, roll center and undercarriage (center), respectively. Subscript in denotes an inertial frame. In this paper we assume that the roll angle ϕ is zero (or known, such that correction is possible).

Only a_y of the above mentioned measurements depends algebraically on v_y. This dependence is given by a force balance in the lateral direction of the vehicle, resulting in

$$a_y = \frac{1}{m} f_y(v_y, u), \tag{14}$$

where f_y is based on a highly nonlinear road/tire friction model and u contains the other known variables/measurements that enter the friction model (see, e.g., Pacejka (2002)). We have here disregarded vehicle roll, and assumed that a_y is the sensor reading of a sensor attached to the vehicle body, aligned with a coordinate system placed in the center of gravity, see Figure 2. This last assumption means that a_y will be different from the actual a_y when there is a non-zero road bank angle (or vehicle roll).

The dynamic equation for v_y (assuming the vehicle is a rigid body, and ignoring roll and pitch motion) is

$$\dot{v}_y = -v_x r + a_y - g \sin\phi_R, \tag{15}$$

where the term $v_x r$ is the rotational term (longitudinal velocity multiplied with yaw rate) stemming from the vehicle body frame being accelerated with respect to the inertial frame. The last term, $g \sin\phi_R$ is non-zero if the road is banked, ϕ_R being the bank angle (see Figure 2) and g the acceleration of gravity. This term can be considered an unmeasured/unknown input ($w = g \sin\phi_R$), and we want to design a (nonlinear) observer for v_y that is robust to non-zero w. This is a challenging problem, see for example the discussion in van Zanten (2000).

For horizontal vehicle models such as (15), and also the bicycle model or the two-track model (Kiencke and Nielsen 2000), any UIO must make use of the a_y measurement. This is because the road bank angle only affects the v_y state, that is, for the model class in Section 3, the only non-zero element of D is the element corresponding to v_y. From (8) we see that we must have $\frac{\partial e}{\partial y}\frac{\partial h}{\partial x}D \neq 0$ for a measurement equation $h(x)$ (the arguments remain the same if h depends on u). This means that we must have $\frac{\partial h}{\partial v_y} \neq 0$. Considering the measurements mentioned above, a_y is the only measurement that depends on v_y and thus h must contain a_y. See also the discussion at the end of Section 2.1.

5.1 *Nonlinear unknown input observer for vehicle lateral velocity on banked roads*

The direct application of the methods in Section 3.2 would lead to an observer that depends on the time-derivative of u, since u appears in the measurement equation (14). This is not desirable, and hence we do not pursue this path. Therefore, we investigate whether it is possible to obtain a linear measurement equation and thus be able to use an observer of the form (6). For this, we must invert the measurement equation (14),

$$v_y^c = f_y^{-1}(a_y, u),$$

where we use v_y^c to distinguish the v_y calculated based on measurements from the real state v_y. This inversion generally requires numerical methods, or we can use an approximation based on a first order Taylor expansion of f_y,

$$v_y^c = f_y^{-1}(a_y, u) \approx \left(\frac{\partial f_y(\hat{v}_y, u)}{\partial v_y}\right)^{-1} \left(ma_y - f_y(\hat{v}_y, u) + \frac{\partial f_y(\hat{v}_y, u)}{\partial v_y}\hat{v}_y\right), \tag{16}$$

where $\hat{v}_y$ is an estimate of v_y, and the closer $\hat{v}_y$ is to v_y, the better the approximation is. Designing a UIO based on this calculated measurement using the results in Section 3.1, we find that we must choose $E = -1$, and get

$$\dot{z} = -Lz, \qquad \hat{v}_y = z + v_y^c.$$

The last equation is implicit in $\hat{v}_y$ if using (16), but this can in principle be solved using standard numerical techniques. However, we see that this observer is nonsensical, z will converge to zero and the estimate will be v_y^c, without any filtration. If we use (9) with $y_1 = v_y^c$ and $y_2 = a_y$ instead, we end up with the following UIO observer:

$$\dot{z} = -L_1 z + L_2\left(a_y - \frac{1}{m}f_y(\hat{v}_y, u)\right), \qquad \hat{v}_y = z + v_y^c. \tag{17}$$

Under the assumption that $\partial f_y/\partial v_y$ has constant sign (see Assumption 5.1 in the next section), it is straightforward to show that this observer is convergent for any $L_1 \geq 0$, and $L_2 > 0$. However, this observer does not use the dynamic model for v_y, since both parts of the observer (z and v_y^c) depend merely on the friction model. This has two drawbacks:

- Measurement noise: All measurement noise (and other measurement errors) will affect the v_y estimate directly. To a certain degree, this problem can be rectified by filtering the estimate. However, better filtering can be obtained by using the dynamic model.
- Measurement model errors: The friction model will contain errors due to assumptions made in specifying the model, tear and wear of tyres, and varying road surfaces. Moreover, for large side-slips, it is practically impossible to invert the friction model, since nonlinear friction models become almost flat with respect to variations in lateral velocity ($\partial f_y/\partial v_y \approx 0$, cf. (16)). The observer (17) has limited possibilities for detuning weight on use of the friction model in such situations.

Therefore, we develop a slightly different observer for this problem, which will not be a UIO by definition, but it is an extension of the observer in Imsland et al. (2006) inspired by the UIO design, taking the above weaknesses into account.

5.2 *Observer with estimation of road bank angle*

Due to the weaknesses of the pure UIO design, we use the UIO observer (17) together with the observer from Imsland et al. (2006), by including an estimate of $w = g \sin \phi_R$:

$$\dot{\hat{v}}_y = -v_x r + a_y - \hat{w} - K_{v_y}(a_y - \frac{1}{m} f_y(\hat{v}_y, u)), \tag{18a}$$

$$\dot{z} = -K_{v_y}\left(a_y - \frac{1}{m} f_y(\hat{v}_y, u)\right), \qquad \nu = z + v_y^c, \tag{18b}$$

$$\hat{w} = K_w(\hat{v}_y - \nu). \tag{18c}$$

We have, for simplicity, chosen $L_1 = 0$, and used $L_2 = K_{v_y}$ for correspondence with the notation in Imsland et al. (2006). Compared to the unknown input observer in the previous section, this observer *does* make use of the dynamic model for v_y, and additionally makes it possible to tune down the effect of $\hat{w}$ on the v_y estimate in periods when we know $\hat{w}$ is inaccurate (that is, in periods when the friction model is inaccurate or not invertible due to for example high slips). Another significant practical advantage over (17), is that use of approximations such as (16) do not result in an implicit equation for $\hat{v}_y$ that in practice must be solved by iteration.

On the other hand, (18) is not a UIO, since the convergence of $\hat{v}_y$ is not independent of w. Its design is, however, strongly inspired by the UIO design (17) that leads to (18b).

This observer is globally convergent under the following two assumptions:

Assumption 5.1 The friction model is continuously differentiable in its arguments, and there exists a constant $c > 0$ such that the following holds globally:

$$\frac{\partial f_y(v_y, u)}{\partial v_y} \leq -c.$$

Assumption 5.2 The road bank angle is constant, $\dot{\phi}_R = 0$.

Assumption 5.1 is realistic, although there may exist combinations of tire models and large slips for which it does not hold on short time intervals, see discussion in Imsland et al. (2006). The second assumption does not hold in general, but it is nevertheless a standard type of assumption in many parameter adaptive state estimation schemes. If the road bank angle changes slowly or rarely, the assumption may be justified. It is notable that this assumption was not needed in the pure UIO design.

Define the observer errors as $\tilde{v}_y := v_y - \hat{v}_y$, and $\tilde{w} := w - \hat{w} = w + K_w(z + \tilde{v}_y)$. The observer error dynamics can then be written, using Assumption 5.2,

$$\dot{\tilde{v}}_y = -\tilde{w} + K_{v_y}\left(a_y - \frac{1}{m} f_y(\hat{v}_y, u)\right), \tag{19a}$$

$$\dot{\tilde{w}} = -K_w \tilde{w}. \tag{19b}$$

Theorem 5.3 *If the observer gains are chosen such that $K_{v_y} K_w > \frac{1}{4mc}$, the origin of the observer error dynamics (19) is uniformly globally exponentially stable (UGES).*

Proof Consider the Lyapunov function candidate $V = \frac{1}{2}(\tilde{v}_y^2 + \tilde{w}^2)$. We have

$$\begin{aligned}\dot{V} &= \tilde{v}_y\left(-\tilde{w} + K_{v_y}(a_y - \frac{1}{m}f_y(\hat{v}_y, u))\right) - K_w\tilde{w}^2 \\ &= \tilde{v}_y K_{v_y}(a_y - \frac{1}{m}f_y(\hat{v}_y, u)) - \tilde{v}_y\tilde{w} - K_w\tilde{w}_y^2 \\ &\leq -mcK_{v_y}\tilde{v}_y^2 - \tilde{v}_y\tilde{w} - K_w\tilde{w}_y^2,\end{aligned}$$

where we have used Assumption 5.1 and the Mean Value Theorem. The right-hand-side expression can be made negative definite by choosing $K_{v_y} > 0$ and $K_w > 0$ such that $K_{v_y}K_w > \frac{1}{4mc}$, and thus V is a Lyapunov function for (19) showing exponential stability (e.g. Khalil (2002)). □

Remark 1 The observer (18) can be extended with yaw rate and longitudinal velocity as estimated states, using a straightforward combination of the above results and those of Imsland et al. (2006).

Remark 2 Note that the above observer is a disturbance observer, which does not depend on time-derivatives of inputs or outputs. The linear disturbance observer in Section 2.2 depends on the time-derivatives of the output, and, in general, this would also be the case if one were to use the same techniques for the class of nonlinear system in Section 3.1. Because of the structure of the problem, this can be avoided in the above design.

Remark 3 One might use v_y^c directly to calculate $\hat{w} = v_y^c - a_y + v_x r$, to insert in (18a). In doing this, however, one does not use feedback from a_y and the friction model to correct errors in v_y^c, that is, one looses a robustifying feedback path.

5.3 *Experimental results from real vehicle measurements*

The real vehicle data are from a (dry concrete surface) test track with straight flat sections and steep banked curves. In the data set illustrated in Figure 3, the car first drives straight, then takes a turn on a banked curve before driving in a slalom pattern on flat ground. As can be seen from the figure, we obtain reasonably good estimates of vehicle velocity. Compared to the estimator without bank angle compensation ((18a) with $\hat{w} = 0$), the velocity estimate is considerably improved during the banked turn. For illustration of the scenario, the individual wheel lateral tire slips are shown in Figure

The estimate of the road bank angle is not very accurate, especially during the slalom driving pattern. The main reasons are model errors, especially due to high side slips during the slalom (discussed below), and roll motion. This vehicle is a relatively large vehicle which exhibits significant roll motion during turning. The effect of roll angle will enter additively to the road bank angle in (15). There is a trade-off here: Tuning down K_w reduces the influence of these errors, but it also increases estimation error when there is non-zero road bank angle.

5.4 *Discussion of lateral velocity observer*

When estimating both v_y and ϕ_R using the measurements explained above, we have two unknowns involved in one dynamic equation (15), and one measurement that depends algebraically on only one of the unknowns (14). Thus, it follows that in estimating both of them, one needs to fully exploit the measurement equation (the friction model). One could argue that adding the dynamic model as in the observer in Section 5.2 does not help, since one is actually adding another unknown (the road bank angle) at the same time. However, it does help if one in addition has information about the dynamics of ϕ_R (for example that it is slowly varying), and other information about the quality of the estimate.

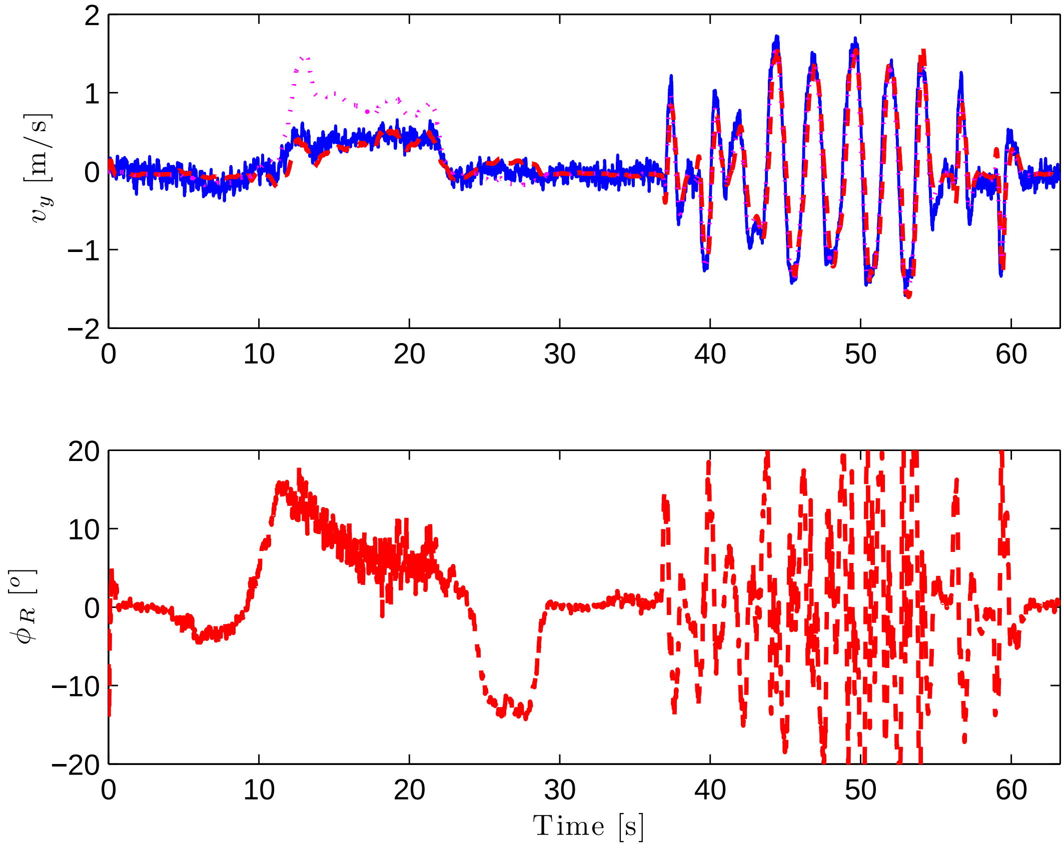

Figure 3. Top: Measured v_y (whole), estimated $\hat{v}_y$ with bank angle compensation (dashed) and estimated $\hat{v}_y$ without bank angle compensation (dotted). Bottom: Estimate of road bank angle. We have used $K_{v_y} = 1$ and $K_w = 8$.

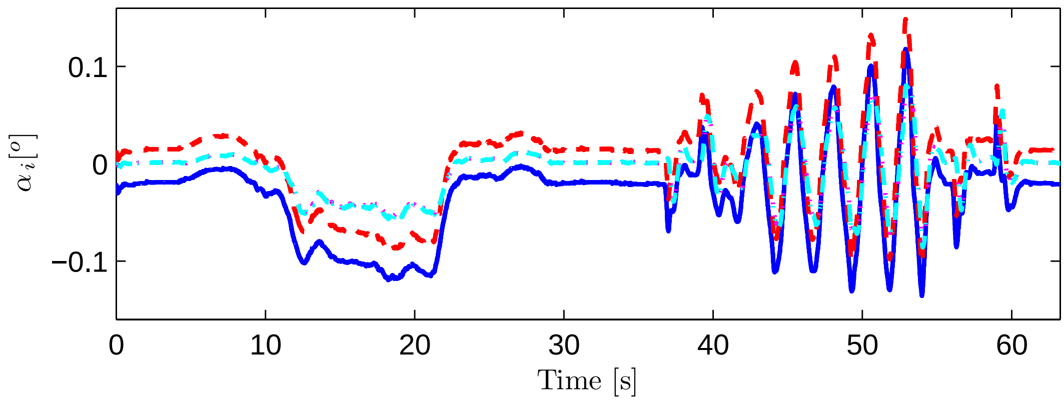

Figure 4. Individual wheel tire slip angles α_i, $i = 1, \ldots, 4$. Whole line is front left, dashed line front right, dotted line rear left, and dash-dotted line rear right (the two rear slip angles are nearly identical).

For example, a problem with the inversion in (16), is that the partial derivative

$$\frac{1}{m}\frac{\partial f_y(v_y, u)}{\partial v_y} \tag{20}$$

may become very small for maneuvers with large side slips, implying that the inverse in (16) becomes large, leading to high gain on a noisy measurement. A possible remedy is to try to recognize this situation and take corrective steps. However, situations where this happen are rather rare and brief, and in most cases, one will drive (also on banked roads) with limited side slips. In the observer used in Figure 3, we set $K_w = 0$ and let $\hat{w}$ approach zero when the estimated partial derivative is below a certain limit. This happens for brief intervals during the slalom-part.

If one uses linear friction models, as in Liu and Peng (2002), the partial derivative (20) will never become small, but the problem remains due to large model errors for large side-slips.

An important parameter in the friction model, the maximum friction coefficient, is known to vary significantly with road conditions. Real-time knowledge of this parameter is therefore important for the observer proposed in the previous section to work. (Grip et al. 2006, 2007) has extended the observer in Imsland et al. (2006) with adaptation of the friction parameter, and our overall goal is to obtain an observer which estimates lateral velocity robustly, subject to different road conditions and road bank angles. As discussed in van Zanten (2000), in some situations it is very hard to distinguish between changes in road bank angle and changes in friction, and hence it may be sensible to only estimate one of them at any given time, using suitable logic.

The complete velocity observer from Imsland et al. (2006), extended with friction adaptation from Grip et al. (2006, 2007) and road bank angle adaptation using the techniques developed here, is implemented, validated and compared to an EKF in Imsland et al. (2007). The results there show that the performance of the nonlinear approach is at least as good as the performance of the EKF, with significantly lower computational complexity.

6 Concluding remarks

A simple observation on the error dynamics of nonlinear unknown input observers has been presented in Theorem 3.1. This observation was first used to specify synthesis inequalities for a class of nonlinear systems with unknown inputs, and then applied to a challenging problem in the automotive industry: Estimation of vehicle lateral velocity on banked roads. The pure UIO approach to this problem proved to have some practical disadvantages, and we therefore, inspired by the UIO design, proposed an approach which better exploits both the dynamic model and the measurement equation.

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References

Arcak, M. and Kokotović, P.: 2001, Nonlinear observers: A circle criterion design and robustness analysis, *Automatica* **37**(12), 1923–1930.

Chen, J., Patton, R. J. and Zhang, H.-Y.: 1996, Design of unknown input observers and robust fault detection filters, *International Journal of Control* **63**(1), 85–105.

Darouach, M., Zasadzinski, M. and Xu, S. J.: 1994, Full-order observers for linear systems with unknown inputs, *IEEE Transactions on Automatic Control* **39**(3), 606–609.

Fan, X. and Arcak, M.: 2003, Observer design for systems with multivariable monotone nonlinearities, *Systems & Control Letters* **50**(4), 319–330.

Fukada, Y.: 1999, Slip-angle estimation for stability control, *Vehicle Systems Dynamics* **32**, 375–388.

Gelb, A. (ed.): 1974, *Applied optimal estimation*, The MIT Press, Cambridge, Mass.-London. Written by the technical staff of The Analytic Sciences Corporation, Principal authors: Arthur Gelb, Joseph F. Kasper, Jr., Raymond A. Nash, Jr., Charles F. Price and Arthur A. Sutherland, Jr.

Grip, H. F., Imsland, L., Johansen, T. A., Fossen, T. I., Kalkkuhl, J. C. and Suissa, A.: 2006, Nonlinear vehicle velocity observer with road-tire friction adaptation, *Proc. 45th IEEE Conf. Decision Contr.*, San Diego.

Grip, H. F., Imsland, L., Johansen, T. A., Fossen, T. I., Kalkkuhl, J. C. and Suissa, A.: 2007, Nonlinear vehicle side-slip estimation with friction adaptation, *Automatica* . Accepted.

Hahn, J.-O., Rajamani, R., You, S.-H. and Lee, K. I.: 2004, Real time identification of road-bank angle using differential GPS, *IEEE Transactions on Control Systems Technology* **12**(4), 589–599.

Imsland, L., Grip, H. F., Johansen, T. A., Fossen, T. I., Kalkkuhl, J. C. and Suissa, A.: 2007, Nonlinear observer for lateral velocity with friction and road bank adaptation – validation and comparison with an extended kalman filter, *Proc. SAE World Congress*.

Imsland, L., Johansen, T. A., Fossen, T. I., Grip, H. F., Kalkkuhl, J. and Suissa, A.: 2006, Vehicle velocity estimation using nonlinear observers, *Automatica* **42**(12), 2091–2103.

Khalil, H. K.: 2002, *Nonlinear Systems*, 3rd edn, Prentice Hall, Upper Saddle River, NJ.

Kiencke, U. and Nielsen, L.: 2000, *Automotive Control Systems*, Springer.

Liu, C.-S. and Peng, H.: 2002, Inverse-dynamics based state and disturbance observers for linear time-invariant systems, *Journal of Dynamic Systems, Measurement, and Control* **124**, 375.

Löfberg, J.: 2004, YALMIP : A toolbox for modeling and optimization in MATLAB, *Proceedings of the CACSD Conference*, Taipei, Taiwan. Available from `http://control.ee.ethz.ch/~joloef/yalmip.php`.

Moreno, J.: 2000, Unknown input observers for SISO nonlinear systems, *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia.

Pacejka, H. B.: 2002, *Tyre and vehicle dynamics*, Butterworth-Heinemann.

Rocha-Cozatl, E., Moreno, J. and Zeitz, M.: 2004, Dissipativity and design of unknown input observers for nonlinear systems, *Proceedings of NOLCOS'04*, Stuttgart, Germany.

Rocha-Cozatl, E., Moreno, J. and Zeitz, M.: 2005, Constructive design of unknown input nonlinear observers by dissipativity and LMIs, *Proceedings of IFAC World Congress*, Prague, Chech republic.

Seliger, R. and Frank, P.: 1991, Fault-diagnosis by disturbance decoupled nonlinear observers, *Proc. 30th IEEE Conf. Decision Contr.*, pp. 2248–2253.

Suissa, A., Zomotor, Z. and Böttiger, F.: 1994, Method for determining variables characterizing vehicle handling. Patent US 5557520.

Tseng, H.: 2001, Dynamic estimation of road bank angle, *Vehicle system dynamics* **36**(4–5), 307–328.

Ungoren, A. Y., Peng, H. and Tseng, H.: 2004, A study on lateral speed estimation methods, *International Journal of Vehicle Autonomous Systems* **2**(1/2), 126–144.

van Zanten, A. T.: 2000, Bosch ESP system: 5 years of experience, *In Proceedings of the Automotive Dynamics & Stability Conference (P-354)*. Paper no. 2000-01-1633.

Vik, B. and Fossen, T. I.: 2001, A nonlinear observer for integration of GPS and INS attitude, *Proc. 40th IEEE Conf. Decision Contr.*, pp. 2956–2961.