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Remarks on the relationship between $\mathcal{L}_p$ stability and internal stability of nonlinear systems

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SUMMARY

In this paper, we investigate the relationship between $\mathcal{L}_p$ stability and internal stability of nonlinear systems. It is shown that under certain conditions, $\mathcal{L}_p$ stability implies attractivity of the equilibrium, and that local $\mathcal{L}_p$ stability with finite gain implies local asymptotic stability of the origin. Copyright © 0000 John Wiley & Sons, Ltd.

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KEY WORDS:

1. INTRODUCTION

In this paper, we study the relationship between $\mathcal{L}_p$ stability and internal stability of nonlinear systems. Specifically, for a nonlinear system that is $\mathcal{L}_p$ stable, we are interested in investigating the internal stability of the autonomous system when the input is zero. The research on this topic evolves mainly along two lines. The first line starts with $\mathcal{L}_p$ stability. An important result that emerges in this direction is [3]. It is shown that under a fairly restrictive condition on the structural property of the system, $\mathcal{L}_p$ stability implies global attractivity of the equilibrium. In fact, it turns out that this conclusion can be attained under much weaker conditions than those in [3]. It is shown in this paper that under mild conditions, global $\mathcal{L}_p$ stability ensures attractivity of the equilibrium in the absence of input and attractivity of the origin with any $\mathcal{L}_p$ input.

The other line emanates from $\mathcal{L}_p$ stability with finite gain. There is a large body of work in the literature in this direction; see, for instance, [2, 6, 1, 3]. Along this line of research, the objective is to conclude local asymptotic stability of the equilibrium based on $\mathcal{L}_p$ stability with finite gain. It was shown in [2] that under a uniform reachability condition, global $\mathcal{L}_p$ stability with finite gain implies local asymptotic stability of the equilibrium. In [6], the notion of small-signal $\mathcal{L}_p$ stability with finite gain was introduced and its connection to attractivity of the equilibrium was established. This concept of small-signal $\mathcal{L}_p$ stability was extended in [1] by so-called gain-over-set stability, and it was shown that finite-gain $\mathcal{L}_p$ stability over a set in $\mathcal{L}_p$ space yields local asymptotic stability

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of the equilibrium. In this paper, we prove a result on the relationship between Lyapunov stability and local $\mathcal{L}_p$ stability with finite gain, which further extends, to some level, the result in [1].

2. PRELIMINARIES

Consider a nonlinear system

$$\Sigma_1 : \quad \dot{x} = f(x, u), \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. We assume that $f(\cdot, u)$ is continuous. Let $x(t, t_0, u, x_0)$ denote the trajectory of Σ_1 initialized at time t_0 with input u and initial condition x_0.

We shall investigate the internal stability of the unforced system

$$\Sigma_2 : \quad \dot{x} = f(x, 0), \quad x(0) = x_0, \tag{2}$$

under the assumption that Σ_1 is $\mathcal{L}_p$ stable in some sense.

We formally define the notions of $\mathcal{L}_p$ stability as follows:

Definition 1
Σ_1 is said to be globally $\mathcal{L}_p$ stable if for $x_0 = 0$ and any $u \in \mathcal{L}_p$, there exists a unique solution $x(\cdot, 0, u, 0) \in \mathcal{L}_p$. Σ_1 is said to be locally $\mathcal{L}_p$ stable with finite gain if there exists a δ and γ such that for $x_0 = 0$ and any u with $\|u\|_{\mathcal{L}_p} \leq \delta$, a unique solution exists and $\|x(\cdot, 0, u, 0)\|_{\mathcal{L}_p} \leq \gamma \|u\|_{\mathcal{L}_p}$.

The domain of attraction and the notion of an $\mathcal{L}_p$-reachable set are defined as follows:

Definition 2
The set

$$\mathcal{A}(\Sigma_2) = \{x_0 \in \mathbb{R}^n \mid x(t, 0, 0, x_0) \to 0 \text{ as } t \to \infty\}$$

is called the domain of attraction of the system Σ_2.

Definition 3
A point $\xi \in \mathbb{R}^n$ is an $\mathcal{L}_p$-reachable point of system Σ_1 if there exist finite T, M and a measurable input $u : [0, T] \to \mathbb{R}^m$ such that $x(T, 0, u, 0) = \xi$ and

$$\int_0^T \|u(t)\|^p \mathrm{d}t \leq M. \tag{3}$$

The set of all $\mathcal{L}_p$-reachable points of Σ_1 is called the $\mathcal{L}_p$-reachable set of Σ_1, which is denoted as $\mathcal{R}_p(\Sigma_1)$.

Remark 1
The requirement (3) in Definition 3 is a weak condition that ensures that the integral of $\|u(t)\|^p$ over the interval $[0, T]$ is finite. For example, any x_0 that is reachable via a signal $u(t)$ that is essentially bounded on $[0, T]$ is $\mathcal{L}_p$-reachable for any $p \in [1, \infty)$.

The following definition of small-signal local $\mathcal{L}_p$-reachability is adapted from [1]:

Definition 4
The system Σ_1 is said to be small-signal locally $\mathcal{L}_p$-reachable if for any $\epsilon > 0$, there exists δ such that for any $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta$, we can find a finite time T and a measurable input $u : [0, T] \to \mathbb{R}^m$ such that $x(T, 0, u, 0) = \xi$ and $\|u\|_{\mathcal{L}_p} \leq \epsilon$.

3. MAIN RESULT

Theorem 1
Suppose system Σ_1 is globally $\mathcal{L}_p$ stable for some $p \in [1, \infty)$. Then $\mathcal{A}(\Sigma_2) \supseteq \mathcal{R}_p(\Sigma_1)$.

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In order to prove Theorem 1, we need the following lemma:

Lemma 1
Consider system Σ_2. If $x(\cdot, 0, 0, x_0) \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then $x(t, 0, 0, x_0) \to 0$.

Proof
For simplicity, we denote $x(t, 0, 0, x_0)$ by $x(t)$ and $f(x(t), 0)$ by $f(x(t))$ in this proof. Suppose, for the sake of establishing a contradiction, that $x(t) \to 0$ does not hold. Then there exists a $\delta > 0$ such that, for any arbitrarily large $T \geq 0$, there is a $\tau \geq T$ such that $\|x(\tau)\| \geq 2\delta$. Let m be a bound on $\|f(x)\|$ on the closed ball $B(2\delta)$. This bound exists due to continuity of $f(x)$ with respect to x.

For some τ such that $\|x(\tau)\| \geq 2\delta$, let $t_2 > \tau$ be the smallest value such that $\|x(t_2)\| = \delta$, and let t_1 be the largest value such that $t_1 < t_2$ and $\|x(t_1)| = 2\delta$. Such t_1 and t_2 exist because $x(t)$ is absolutely continuous and $x \in \mathcal{L}_p$. Since $\|x(t)\| \in B(2\delta)$ for all $t \in [t_1, t_2]$, we have, due to the absolute continuity of the solution,

$$\|x(t_1)\| - \|x(t_2)\| \leq \|x(t_2) - x(t_1)\| = \left\| \int_{t_1}^{t_2} f(x(\tau))\, \mathrm{d}\tau \right\| \leq \int_{t_1}^{t_2} \|f(x(\tau))\|\, \mathrm{d}\tau \leq (t_2 - t_1)m.$$

Hence, $t_2 - t_1 \geq (\|x(t_1)\| - \|x(t_2)\|)/m = \delta/m$. Clearly $\|x(t)\| \geq \delta$ for all $t \in [\tau, t_2]$, and furthermore $t_2 - \tau \geq t_2 - t_1 \geq \delta/m$. It follows that for each τ such that $\|x(\tau)\| \geq 2\delta$, we have $\|x(t)\| \geq \delta$ for all $t \in [\tau, \tau + \delta/m]$.

Let T be chosen large enough that

$$\int_T^\infty \|x(t)\|^p\, \mathrm{d}\tau < \frac{\delta^{p+1}}{m}. \tag{4}$$

Such a T must exist, since $x(t) \in \mathcal{L}_p$. Let $\tau \geq T$ be chosen such that $\|x(\tau)\| \geq 2\delta$. We have

$$\int_T^\infty \|x(t)\|^p\, \mathrm{d}\tau \geq \int_\tau^{\tau+\delta/m} \|x(t)\|^p\, \mathrm{d}\tau \geq \frac{\delta^{p+1}}{m}.$$

This contradicts (4), which proves that $x(t) \to 0$. □

Remark 2
As pointed out to us by one of the reviewers of this paper, the result in Lemma 1 is closely connected to Theorem 1 in [5]. The proof given above also employs a similar computation technique as used in [5]. Here we are only concerned about attractivity of the equilibrium whereas in Theorem 1 of [5], a result of global asymptotic stability was proved. Therefore, only a weaker condition that $x(\cdot, 0, 0, x_0) \in \mathcal{L}_p$ is required compared with [5] where the $\mathcal{L}_p$ norm of the trajectory needs to be a class $\mathcal{K}$ function of the $\|x_0\|$.

Proof of Theorem 1
For any $x_0 \in \mathcal{R}_p(\Sigma_1)$, there exist finite T, M and an input $u_0(t)$ for $t \in [0, T]$ such that $x(T, 0, u_0, 0) = x_0$ and

$$\int_0^T \|u_0(t)\|^p \mathrm{d}t \leq M$$

Define

$$u(t) = \begin{cases} u_0(t), & t \in [0, T] \\ 0, & t > T \end{cases}$$

Clearly, $u \in \mathcal{L}_p$. Since Σ_1 is globally $\mathcal{L}_p$ stable, we have that $x(\cdot, 0, u, 0) \in \mathcal{L}_p$. On the other hand, $u(t) = 0$ for $t > T$ implies that after T the system Σ_1 is equivalent with system Σ_2 initialized at x_0, i.e. $x(t, 0, u, 0) = x(t - T, 0, 0, x_0)$ with $t > T$. Therefore, $x(t, 0, 0, x_0) \in \mathcal{L}_p$ over $[0, \infty)$. It follows from Lemma 1 that $x(t, 0, 0, x_0) \to 0$ as $t \to \infty$. This completes the proof. □

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Corollary 1
Suppose system Σ_1 is globally $\mathcal{L}_p$ stable for some $p \in [1, \infty)$. If $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$, then the origin of Σ_2 is globally attractive.

The next theorem shows that under a certain condition on the structure of $f(x, u)$, the origin of Σ_1 is attractive for any input $u \in \mathcal{L}_p$.

Theorem 2
Suppose that Σ_1 is globally $\mathcal{L}_p$ stable for some $p \in [1, \infty)$. If there exist $\delta > 0$, $m_1 \geq 0$, $m_2 \geq 0$ and $q \in [0, p]$ such that for any x with $\|x\| \leq \delta$

$$\|f(x, u)\| \leq m_1 + m_2 \|u\|^q, \tag{5}$$

then for $x_0 = 0$ and any $u \in \mathcal{L}_p$, $x(t, 0, u, 0) \to 0$ as $t \to \infty$.

Proof
Define a generalized saturation function $\bar{\sigma}(\cdot) : \mathcal{R}^n \to \mathcal{R}^n \in C^1$ as

$$\bar{\sigma}(x) = \begin{bmatrix} \bar{\sigma}_1(x_1) \\ \vdots \\ \bar{\sigma}_n(x_n) \end{bmatrix}, \quad \bar{\sigma}_i(x_i) = \begin{cases} -\frac{2\delta}{\pi}, & x_i < -\delta \\ \frac{2\delta}{\pi} \sin(\frac{\pi}{2\delta} x_i), & |x_i| \leq \delta \\ \frac{2\delta}{\pi}, & x_i > \delta \end{cases}$$

Consider $\bar{x}(t) = \bar{\sigma}(x(t, 0, u, 0))$. Note that $\bar{x}(t)$ is still absolutely continuous on any compact interval. Let $\bar{x}_i$ and f_i denote the ith element of $\bar{x}$ and $f(x, u)$ respectively. We have

$$|\dot{\bar{x}}_i(t)| = \begin{cases} 0, & |x_i(t)| > \delta \\ |\cos(\frac{\pi}{2\delta} x_i) f_i(x(t), u(t))| \leq m_1 + m_2 \|u(t)\|^q, & |x_i(t)| \leq \delta \end{cases}$$

Therefore, $\|\dot{\bar{x}}(t)\| \leq \sqrt{n}(m_1 + m_2 \|u\|^q)$ for all $t > 0$. Note that $\|u(t)\|^q \leq 1 + \|u(t)\|^p$ and hence $\|u\|^q$ is locally uniformly integrable. Then it follows from [4] that $\bar{x}(t) \to 0$ as $t \to 0$. This implies that $x(t, 0, u, 0) \to 0$ as $t \to 0$. □

Remark 3
In [3], in order to prove the same result as in Theorem 2, the following condition was imposed on $f(x, u)$: there exists δ_1, K_1, K_2 and $\alpha \in [0, p]$ such that for $x \in \mathbb{R}^n$ with $\|x\| \leq \delta_1$,

$$\|f(x, u)\| \leq K_1(\|x\| + \|u\|) + K_2(\|x\|^\alpha + \|u\|^\alpha)$$

Theorem 2 shows that the restrictions on x in the above condition are not necessary.

An immediate consequence of Theorem 2 is the next theorem.

Theorem 3
Suppose that Σ_1 is globally $\mathcal{L}_p$ stable and $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$ for some $p \in [1, \infty)$. If there exist $\delta > 0$, $m_1 \geq 0$, $m_2 \geq 0$ and $q \in [0, p]$ such that for any x with $\|x\| \leq \delta$

$$\|f(x, u)\| \leq m_1 + m_2 \|u\|^q,$$

then Σ_1 is globally $\mathcal{L}_p$ stable with arbitrary initial condition.† Moreover, for any $x_0 \in \mathbb{R}^n$ and any $u \in \mathcal{L}_p$, $x(t, 0, u, x_0) \to 0$ as $t \to \infty$.

Proof
Since $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$, for any $x_0 \in \mathbb{R}^n$, there exist finite T, M and a measurable input $u_0 : [0, T] \to$

† Σ_1 is said to be global $\mathcal{L}_p$ stable with arbitrary initial condition if for any $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{L}_p$, we have $x(\cdot, 0, u, x_0) \in \mathcal{L}_p$.

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$\mathbb{R}^m$ such that $x(T, 0, u_0, 0) = x_0$ and

$$\int_0^T \|u_0(t)\|^p \, \mathrm{d}t \le M.$$

For any $u \in \mathcal{L}_p$, define

$$\bar{u}(t) = \begin{cases} u_0(t), & t \in [0, T] \\ u(t - T), & t > T \end{cases}$$

Then we have $x(t, 0, u, x_0) = x(t + T, 0, \bar{u}, 0)$. Clearly $\bar{u} \in \mathcal{L}_p$. This implies that $x(\cdot, 0, \bar{u}, 0) \in \mathcal{L}_p$ and hence $x(\cdot, 0, u, x_0) \in \mathcal{L}_p$. This proves $\mathcal{L}_p$ stability with arbitrary initial condition and it follows from Theorem 2 that $x(t, 0, \bar{u}, 0) \to 0$ as $t \to \infty$ and therefore $x(t, 0, u, x_0) \to 0$ as $t \to \infty$. □

In what follows, we prove a theorem that is a slight generalization of results in [1].

Theorem 4
Suppose that Σ_1 is locally $\mathcal{L}_p$ stable with finite gain and small-signal locally $\mathcal{L}_p$-reachable. Then the origin of Σ_2 is locally asymptotically stable.

Proof
Let ϵ be an arbitrary positive real number. We need to show that there exists a $\delta > 0$ such that $\|x_0\| \le \delta$ implies $\|x(t, 0, 0, x_0)\| \le \varepsilon$ for all $t \ge 0$. Toward this end, let $\delta \le \frac{\epsilon}{2}$ be chosen such that for any $x_0 \in \mathbb{R}^n$ with $\|x_0\| \le \delta$, there exist a finite time T and measurable input $u : [0, T] \to \mathbb{R}^m$ such that

$$x(T, 0, u, 0) = x_0 \text{ and } \|u\|_{\mathcal{L}_p} < \tfrac{\epsilon}{2\gamma} \left(\tfrac{\epsilon}{2M(\epsilon)}\right)^{\frac{1}{p}}.$$

This is possible due to $\mathcal{L}_p$ local reachability.

Set $u(t) = 0$ for $t > T$. Since Σ_1 is locally $\mathcal{L}_p$ stable with finite gain, from Definition 1, there exists γ such that

$$\int_T^\infty \|x(t, 0, u, 0)\|^p \, \mathrm{d}t \le \int_0^\infty \|x(t, 0, u, 0)\|^p \, \mathrm{d}t \le \gamma^p \|u\|^p_{\mathcal{L}_p} < \tfrac{\epsilon^{p+1}}{2^{p+1} M(\epsilon)}.$$

For $t > T$, the system Σ_1 is equivalent to Σ_2 initialized at $x(0) = x_0$, i.e. $x(t, 0, u, 0) = x(t - T, 0, 0, x_0)$. Hence we have

$$\int_0^\infty \|x(t, 0, 0, x_0)\|^p \, \mathrm{d}t < \tfrac{\epsilon^{p+1}}{2^{p+1} M(\epsilon)}. \tag{6}$$

It immediately follows from Lemma 1 that $x(t, 0, 0, x_0) \to 0$ as $t \to \infty$.

We proceed to show that $\|x(t, 0, 0, x_0)\| < \epsilon$ for all $t \ge 0$. Suppose, for the sake of establishing a contradiction, that there exists a τ such that $\|x(\tau, 0, 0, x_0)\| \ge \epsilon$. Let $t_1 < \tau$ be the largest value such that $\|x(t_1, 0, 0, x_0)\| = \epsilon/2$, and let $t_2 \le \tau$ be the smallest value such that $t_2 > t_1$ and $\|x(t_2, 0, 0, x_0)\| = \epsilon$. Such t_1 and t_2 exist because $\|x_0\| \le \frac{\epsilon}{2}$. Then $\epsilon/2 \le \|x(t, 0, 0, x_0)\| \le \epsilon$ for all $t \in [t_1, t_2]$. Let $M(\epsilon)$ be a bound on $f(x, 0)$ for $\|x\| \le \epsilon$. We have, owing to the absolute continuity of $x(\cdot, 0, 0, x_0)$,

$$\begin{aligned} \|x(t_2, 0, 0, x_0)\| - \|x(t_1, 0, 0, x_0)\| &\le \|x(t_2, 0, 0, x_0) - x(t_1, 0, 0, x_0)\| \\ &\le \left\| \int_{t_1}^{t_2} f(x(t), 0) \, \mathrm{d}t \right\| \le \int_{t_1}^{t_2} M(\epsilon) \, \mathrm{d}t \le M(\epsilon)(t_2 - t_1) \end{aligned}$$

This gives that $t_2 - t_1 \ge \frac{\epsilon}{2M(\epsilon)}$ and hence that

$$\int_0^\infty \|x(t, 0, 0, x_0)\|^p \, \mathrm{d}t \ge \int_{t_1}^{t_2} \|x(t, 0, 0, x_0)\|^p \, \mathrm{d}t \ge \int_{t_1}^{t_2} \left(\frac{\epsilon}{2}\right)^p \mathrm{d}t = \tfrac{\epsilon^{p+1}}{2^{p+1} M(\epsilon)},$$

which contradicts (6). Hence $\|x(t, 0, 0, x_0)\| < \epsilon$ for all $t \ge 0$, which completes the proof. □

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Remark 4
Compared with the result in [1], Theorem 4 requires a finite gain only within an arbitrary small neighborhood of the origin of $\mathcal{L}_p$ space.

Remark 5
We assume in this paper that $f(x, u)$ is continuous with respect to x, which covers a large class of dynamical systems. In fact, it can be seen from the proof that we only need continuity of $f(x, u)$ with respect to x at $x = 0$.

Remark 6
We also refer the reader to a related paper [5] which presents the integral characterizations of uniform asymptotic stability and uniform exponential stability for differential equations and inclusions.

4. CONCLUSION

In this paper, we study the connection between two notions of stability of nonlinear systems, namely Lyapunov stability and external $\mathcal{L}_p$ stability. While no direct translation can be made between these notions of stability in general, this paper represents another effort in exposing the relationship between them by studying attractivity for $\mathcal{L}_p$ stable systems with additional structural properties, such as local reachability and bounds on the derivative.

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