

# Further results on the disturbance response of a double integrator controlled by saturating linear static state feedback

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## Abstract

In this paper we study the response to external disturbances of a double integrator with saturating feedback. For a class of disturbances that have bounded integrals over all intervals, we show that a linear static feedback law can always be designed to ensure boundedness of the states. Moreover, boundedness can be preserved if the disturbance is biased by a small DC signal. In the special case that the disturbance is made up of a finite number of sinusoids and a small bias, any linear static feedback ensures boundedness of the states. These results are an extension of previous results by Wen, Roy, and Saberi (2008).

*Key words:* Double integrator, input saturation, external disturbances.

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## 1 Introduction

The double integrator system is commonly seen in control applications including low-friction, free rigid-body motion, such as single-axis spacecraft rotation and rotary crane motion (see Rao and Bernstein, 2001, and references therein). Of particular interest is the control of double integrators subject to input saturation. A classical result is that a double integrator with a saturating linear static feedback provides global asymptotic stability of the origin. This result has been extended to mixed-type systems by Tyan and Bernstein (1999) and Yang, Stoorvogel, and Saberi (2010). Many other control methods have also been proposed. A brief summary and comparison of various methods is given by Rao and Bernstein (2001).

Compared with the relatively mature study of internal stabilization, the dynamic response of a double integrator with saturating feedback to external disturbances is still not fully understood. In this paper, we study the disturbance response

of a double integrator controlled by a saturating linear static state feedback, as given below:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sigma(-k_1 x_1 - k_2 x_2) + d.\end{aligned}\tag{1}$$

where  $\sigma$  represents the standard saturation function  $\sigma(s) = \text{sign}(s) \min\{1, |s|\}$ . The goal is to identify a class of disturbances for which the states of the above controlled system remain bounded. Chitour (2001) and Stoorvogel, Shi, and Saberi (2004) have previously studied this problem in the context of  $\mathcal{L}_p$  stability. They showed that for any  $k_1 > 0$  and  $k_2 > 0$ , (1) is  $\mathcal{L}_p$  stable for  $p \in [1, 2]$ , and, moreover, the trajectories remain bounded for any  $d \in \mathcal{L}_p$  with  $p \in [1, 2]$ . However, for any  $p > 2$  there exists  $d \in \mathcal{L}_p$  that cause the states to grow unbounded from certain initial conditions. Shi and Saberi (2002) studied the notion of input-to-state stability (ISS) (see Sontag and Wang, 1995) for system (1), and proved that no choice of  $k_1, k_2$  can achieve ISS. Specifically, there exist bounded disturbances with arbitrarily small  $\mathcal{L}_\infty$  norm that cause the states to grow unbounded from certain initial conditions. Even more dramatically, unbounded growth can be achieved by *vanishing* disturbances with arbitrarily small  $\mathcal{L}_\infty$  norm.

Wen et al. (2008) extended the negative result from Shi and Saberi (2002) to classes of small  $\mathcal{L}_\infty$  signals with further restrictions. However, Wen et al. (2008) also showed that boundedness of the states of (1) is preserved for a particular

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class of small disturbances that have bounded integrals over all intervals. To be precise, let a family of parameterized sets be defined by

$$\tilde{\Omega}_M = \left\{ d \in \mathcal{L}_\infty(1) \mid \forall t_1, t_2 \geq 0, \left| \int_{t_1}^{t_2} d(t) dt \right| \leq M \right\},$$

where  $\mathcal{L}_\infty(1)$  denotes the set of  $\mathcal{L}_\infty$  signals of magnitude less than 1. For a given linear state feedback, it was proven that for any  $M > 0$  there exists a  $q^* > 0$  such that the states of (1) remain bounded for any disturbance  $d(t) = qs(t)$ , where  $s(t) \in \tilde{\Omega}_M$  and  $q \leq q^*$ . In other words, signals with bounded integrals can be tolerated if they are scaled down by a sufficient amount. It was furthermore shown that a small bias in  $d(t)$  can be tolerated while still achieving boundedness of the states. The class of disturbances considered by Wen et al. (2008) covers a broad class of signals, such as periodic, quasi-periodic, and  $\mathcal{L}_1$  signals.

This paper is an extension of the work in Wen et al. (2008). Our focus is also on disturbances with bounded integrals; however, we strengthen the results from Wen et al. (2008) by dispensing with the magnitude restriction on  $d$  and removing the attenuation factor  $q$ . Specifically, we consider the family of parameterized sets

$$\Omega_M = \left\{ d \in \mathcal{L}_\infty \mid \forall t_1, t_2 \geq 0, \left| \int_{t_1}^{t_2} d(t) dt \right| \leq M \right\}. \quad (2)$$

If a signal  $d$  belongs to  $\Omega_M$  for some  $M > 0$ , we refer to it as an *integral-bounded signal*, and we refer to  $M$  as an *integral bound* on  $d$ .

We first prove a new negative result, namely, that for a given linear static feedback law, there always exist integral-bounded signals that cause the trajectories of the system to grow unbounded from some initial conditions. Our next result, however, shows that if an integral bound  $M$  is known *a priori*, then  $k_1$  and  $k_2$  can always be designed to ensure boundedness of the states, regardless of initial conditions. Moreover, boundedness can be ensured also if the integral-bounded disturbance is biased by a DC signal of magnitude less than 1. Finally, we prove an even stronger result for disturbances consisting of a finite number of sinusoids plus a DC offset of magnitude less than 1. In this case, *any* internally stabilizing linear static feedback law ensures boundedness of the states.

## 2 Main result

The first theorem shows that not every internally stabilizing static linear law can maintain boundedness of the trajectories in the face of integral-bounded disturbances.

**Theorem 1** *Consider the system (1) with  $k_1 > 0$  and  $k_2 > 0$ . There exists an integral-bounded signal  $d$  and an initial condition such that  $x_1$  and  $x_2$  grow unbounded.*

**Proof.** Define  $y_1 = k_1 x_1 + k_2 x_2$ ,  $y_2 = k_2 x_2$  and  $\tilde{t} = \frac{k_1}{k_2} t$ . Then the closed-loop system in the new coordinates and with  $\tilde{t}$  as the time variable becomes

$$\begin{aligned} \frac{dy_1}{d\tilde{t}}(\tilde{t}) &= y_2(\tilde{t}) - \lambda [\sigma(y_1(\tilde{t})) - d(\tilde{t})], \\ \frac{dy_2}{d\tilde{t}}(\tilde{t}) &= -\lambda [\sigma(y_1(\tilde{t})) - d(\tilde{t})], \end{aligned} \quad (3)$$

where  $\lambda = k_2^2/k_1 > 0$ . We shall construct an integral-bounded disturbance  $d$  that causes the states to grow unbounded from a particular initial condition.

**Step 1:** Suppose the trajectory of (3) starts from  $A = (1, N\lambda)$ , for some large integer  $N$ , at time  $\tilde{t}_A = 0$ . We will construct a  $d$  to drive the states from point  $A$  to  $B = (1, -(N+2)\lambda)$  at time  $\tilde{t}_B$ . Choose

$$d(\tilde{t}) = 2\pi \sin(\pi\tilde{t}).$$

Let

$$\begin{aligned} d_1(\tilde{t}) &= \int_0^{\tilde{t}} d(\tau) d\tau = 2(1 - \cos(\pi\tilde{t})), \\ d_2(\tilde{t}) &= \int_0^{\tilde{t}} d_1(\tau) d\tau = 2\tilde{t} - \frac{2}{\pi} \sin(\pi\tilde{t}). \end{aligned}$$

Since  $\frac{dy_1}{d\tilde{t}}(0) = y_2(0) - \lambda$ , we see that for large  $N$ , the trajectory will initially move to the right. If  $y_1(\tilde{t}) > 1$ , we have

$$y_2(\tilde{t}) = y_2(0) - \lambda\tilde{t} + \lambda d_1(\tilde{t}) = N\lambda - \lambda\tilde{t} + 2\lambda(1 - \cos(\pi\tilde{t}))$$

and

$$\begin{aligned} y_1(\tilde{t}) &= y_1(0) + \int_0^{\tilde{t}} y_2(\tau) d\tau - \lambda\tilde{t} + \lambda d_1(\tilde{t}) \\ &= y_1(0) + N\lambda\tilde{t} - \frac{\lambda}{2}\tilde{t}^2 + \lambda d_2(\tilde{t}) - \lambda\tilde{t} + \lambda d_1(\tilde{t}) \\ &= y_1(0) + N\lambda\tilde{t} - \frac{\lambda}{2}\tilde{t}^2 + 2\lambda\tilde{t} - \frac{2\lambda}{\pi} \sin(\pi\tilde{t}) - \lambda\tilde{t} \\ &\quad + 2\lambda(1 - \cos(\pi\tilde{t})) \\ &= -\frac{\lambda}{2}\tilde{t}^2 + (N\lambda + \lambda)\tilde{t} + 2\lambda(1 - \cos(\pi\tilde{t})) \\ &\quad - \frac{2\lambda}{\pi} \sin(\pi\tilde{t}) + 1. \end{aligned}$$

Given a sufficiently large  $N$ ,  $y_1(\tilde{t})$  only has one intersection with  $y_1 = 1$  for  $\tilde{t} > 0$ . This is shown in the Appendix. For  $\tilde{t} = 2N+2$ , we have

$$y_1 = -\lambda \frac{(2N+2)^2}{2} + (N\lambda + \lambda) \times (2N+2) + 1 = 1$$

and

$$y_2 = N\lambda - \lambda(2N+2) + \lambda \int_0^{2N+2} d(\tilde{t}) d\tilde{t} = -(N+2)\lambda.$$

This shows that the trajectory will cross  $y_1 = 1$  at  $B = (1, -(N+2)\lambda)$  at time  $\tilde{t}_B = 2N+2$ . We have

$$\int_0^{\tilde{t}_B} d(\tilde{t}) d\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t}) d\tilde{t} \right| \leq 4 \text{ with } \tilde{t}_A \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_B.$$

**Step 2:** From  $\tilde{t}_B$ , set  $d(\tilde{t}) = \sigma(y_1(\tilde{t}))$ . Then

$$\begin{aligned} \frac{dy_1}{d\tilde{t}}(\tilde{t}) &= y_2, \\ \frac{dy_2}{d\tilde{t}}(\tilde{t}) &= 0. \end{aligned}$$

The trajectory will move directly toward the left to  $C = (-1, -(N+2)\lambda)$  at time  $\tilde{t}_C = \tilde{t}_B + \frac{2}{(N+2)\lambda}$ . Clearly, for  $N$  sufficiently large,

$$\int_{\tilde{t}_B}^{\tilde{t}_C} d(\tilde{t}) d\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t}) d\tilde{t} \right| \leq 1 \text{ with } \tilde{t}_B \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_C.$$

**Step 3:** From  $\tilde{t}_C$ , choose

$$d(\tilde{t}) = -2\pi \sin(\pi\tilde{t}).$$

Following the same argument as in Step 1 with  $N$  replaced by  $\tilde{N} = N+2$ , we find that the trajectory will re-cross  $y_1 = -1$  at  $D = (-1, (\tilde{N}+2)\lambda) = (-1, (N+4)\lambda)$  at time  $\tilde{t}_D = \tilde{t}_C + 2\tilde{N} + 2 = \tilde{t}_C + 2N + 6$ . Similarly

$$\int_{\tilde{t}_C}^{\tilde{t}_D} d(\tilde{t}) d\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t}) d\tilde{t} \right| \leq 4 \text{ with } \tilde{t}_C \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_D.$$

**Step 4:** From  $\tilde{t}_D$ , choose

$$d(\tilde{t}) = \sigma(y_1(\tilde{t})).$$

The trajectory will move directly to the right and cross  $y_1 = 1$  at  $E = (1, (N+4)\lambda)$  at time  $\tilde{t}_E = \tilde{t}_D + \frac{2}{N+4}$ . We have

$$\int_{\tilde{t}_D}^{\tilde{t}_E} d(\tilde{t}) d\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t}) d\tilde{t} \right| \leq 1 \text{ with } \tilde{t}_D \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_E.$$

The system trajectory resulting from Steps 1 through 4 is visualized in Fig. 1. By repeating these steps, the state grows unbounded, and we can check that the constructed disturbance signal satisfies

$$\left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t}) d\tilde{t} \right| \leq 4 + 1 + 1 + 4 = 10 = \tilde{M}$$

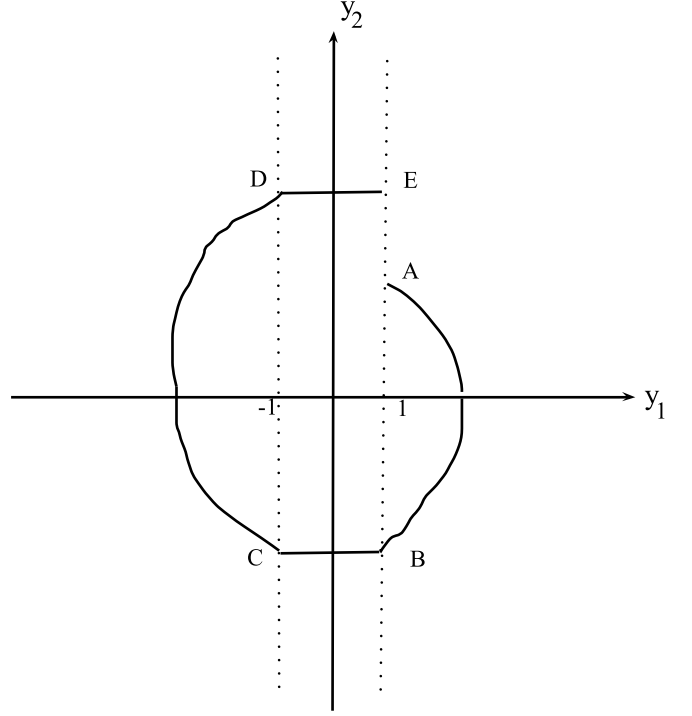


Fig. 1. State trajectories of double integrator with  $\lambda = 1$

for any  $0 = \tilde{t}_A \leq \tilde{t}_1 \leq \tilde{t}_2$ . In the original time variable  $t$ , we calculate the integral bound as

$$\left| \int_{t_1}^{t_2} d(\tilde{t}) dt \right| = \left| \frac{k_2}{k_1} \int_{\frac{k_1}{k_2} t_1}^{\frac{k_1}{k_2} t_2} d(\tilde{t}) d\tilde{t} \right| \leq \frac{k_2}{k_1} \tilde{M}.$$

Hence, we have shown that there exists an integral-bounded signal with integral bound  $M = \frac{k_2}{k_1} \tilde{M} = 10 \frac{k_2}{k_1}$  that causes the states to grow unbounded for a particular initial condition. ■

Next, we show that if an integral bound  $M$  is given *a priori*, then we can always design a static stabilizing linear feedback to ensure boundedness of the trajectories.

**Theorem 2** *Let  $M$  be given. If  $k_1$  and  $k_2$  satisfy  $\frac{k_2}{k_1} > 16M$ , then for any  $d \in \Omega_M$  and any initial condition, we have  $x_1, x_2 \in \mathcal{L}_\infty$ .*

**Proof.** The proof of Theorem 2 is a consequence of Lemmas 3 and 4 which are stated and proved below.

**Lemma 3** *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_2 + y, \\ \dot{x}_2 &= \sigma(-k_1 x_1 - k_2 x_2), \end{aligned} \quad (4)$$

where  $|y(t)| < 2M$  for all  $t \geq 0$  and  $\frac{k_2}{k_1} > 16M$ . In that case, we have  $x_1, x_2 \in \mathcal{L}_\infty$  for any initial condition.

**Proof of Lemma 3.** Define a positive definite function  $V$  as

$$V = \int_0^{k_1 x_1} \sigma(s) ds + \int_0^{k_1 x_1 + k_2 x_2} \sigma(s) ds + k_1 x_2^2.$$

This function was first introduced in Chitour (2001). Differentiating  $V$  along the trajectories yields

$$\begin{aligned} \dot{V} &= (k_1 x_2 + k_1 y) \sigma(k_1 x_1) - 2k_1 x_2 \sigma(k_1 x_1 + k_2 x_2) \\ &\quad + [k_1 x_2 + k_1 y - k_2 \sigma(k_1 x_1 + k_2 x_2)] \sigma(k_1 x_1 + k_2 x_2) \\ &= k_1 x_2 [\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] - k_2 \sigma^2(k_1 x_1 + k_2 x_2) \\ &\quad + k_1 y [\sigma(k_1 x_1 + k_2 x_2) + \sigma(k_1 x_1)] \\ &\leq k_1 x_2 [\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] - k_2 \sigma^2(k_1 x_1 + k_2 x_2) \\ &\quad + 2k_1 |y|. \end{aligned}$$

If  $|k_1 x_1 + k_2 x_2| > \frac{1}{2}$ , then

$$-k_2 \sigma^2(k_1 x_1 + k_2 x_2) + 2k_1 |y| \leq -16Mk_1 \times \frac{1}{4} + 4k_1 M \leq 0.$$

Hence

$$\dot{V} \leq k_1 x_2 [\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] \leq 0.$$

If  $|k_1 x_1 + k_2 x_2| \leq \frac{1}{2}$ , then by using Lemma B.2 in Shi, Saberi, and Stoorvogel (2003), we get

$$k_1 x_2 [\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] \leq -\frac{k_1}{2} x_2 \sigma(k_2 x_2).$$

If we also have that  $|x_2| \geq \max\{8M, \frac{1}{k_2}\}$ , then

$$\begin{aligned} k_1 x_2 [\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] \\ \leq -\frac{k_1}{2} x_2 \sigma(k_2 x_2) \leq -4k_1 M, \end{aligned}$$

which yields  $\dot{V} \leq 0$ . We therefore conclude that  $\dot{V} \leq 0$  outside the region defined by  $|k_1 x_1 + k_2 x_2| \leq \frac{1}{2}$  and  $|x_2| \leq \max\{8M, \frac{1}{k_2}\}$ . Hence,  $V$  remains bounded, which implies that  $x_1, x_2 \in \mathcal{L}_\infty$ . ■

Now consider the double integrator system (1). We construct a fictitious state

$$\begin{aligned} \dot{y} &= \sigma(-k_1 x_1 - k_2 x_2) - \sigma(-k_1 x_1 - k_2 x_2 + k_2 y) + d, \\ y(0) &= 0. \end{aligned}$$

By defining  $z = x_2 - y$ , we obtain the augmented system

$$\begin{aligned} \dot{x}_1 &= z + y, \\ \dot{z} &= \sigma(-k_1 x_1 - k_2 z), \\ \dot{y} &= \sigma(-k_1 x_1 - k_2 x_2) - \sigma(-k_1 x_1 - k_2 x_2 + k_2 y) + d, \end{aligned}$$

with  $y(0) = 0, z(0) = x_2(0)$ . From Lemma 3, we know that given  $\frac{k_2}{k_1} > 16M$ ,  $x_1$  and  $z$  remain bounded provided  $|y| \leq 2M$ . The latter statement is proven by the following lemma.

**Lemma 4** Consider the system

$$\dot{y} = \sigma(v) - \sigma(v + k_2 y) + d, \quad y(0) = 0, \quad (5)$$

where  $k_2 > 0, d \in \Omega_M$  and  $v$  is continuous. We have  $|y(t)| \leq 2M$  for all  $t \geq 0$ .

**Proof of Lemma 4.** Define

$$\dot{\bar{y}} = d, \quad \bar{y}(0) = 0.$$

Since  $d \in \Omega_M$ , the solution satisfies  $|\bar{y}(t)| \leq M$  for all  $t \geq 0$ . Define  $\tilde{y} = y - \bar{y}$ . We have

$$\dot{\tilde{y}} = \sigma(v) - \sigma(v + k_2(\tilde{y} + \bar{y})), \quad \tilde{y}(0) = 0.$$

Define a positive definite function  $\tilde{V} = \tilde{y}^2$ . Taking the derivative of  $\tilde{V}$  with respect to  $t$ , we get

$$\dot{\tilde{V}} = \tilde{y} [\sigma(v) - \sigma(v + k_2(\tilde{y} + \bar{y}))].$$

If  $\tilde{V} \geq M^2$ , then  $|\tilde{y}(t)| \geq M \geq |\bar{y}(t)|$ , which implies that  $k_2(\tilde{y} + \bar{y})$  has the same sign as  $\tilde{y}$ . It then follows that  $\dot{\tilde{V}} \leq 0$ . Since  $\tilde{V}(0) = 0$ , we can conclude that  $\tilde{V} \leq M^2$  and thus  $|\tilde{y}(t)| \leq M$  for all  $t \geq 0$ , and it follows that  $|y(t)| \leq |\bar{y}(t)| + |\tilde{y}(t)| \leq 2M$  for all  $t \geq 0$ . ■

From Lemmas 3 and 4, we know that  $x_1$  and  $z$  are bounded. Since  $y$  is bounded as shown in Lemma 4, we conclude that  $x_1, x_2 \in \mathcal{L}_\infty$ . ■

An immediate consequence of Theorems 1 and 2 is that if  $k_1$  and  $k_2$  are arbitrary positive real numbers, then boundedness is guaranteed if the integral bound  $M$  is sufficiently small. This is formally stated in the following corollary.

**Corollary 5** For any given  $k_1 > 0$  and  $k_2 > 0$ , we have  $x_1, x_2 \in \mathcal{L}_\infty$  if  $d \in \Omega_M$  with  $M \leq \frac{k_2}{16k_1}$ .

In the next theorem we consider integral-bounded disturbances that are biased by a DC signal. We show that, if the magnitude of the bias is less than 1 by a known margin, and an integral bound  $M$  is known *a priori*, then  $k_1, k_2$  can be chosen to ensure boundedness of  $x_1, x_2$ .

**Theorem 6** Let  $M > 0$  and  $\delta \in (0, 1]$  be given, and suppose that  $d = d_1 + d_2$  where  $d_1$  is a constant with  $|d_1| \leq 1 - \delta$  and  $d_2 \in \Omega_M$ . If  $k_1, k_2$  satisfy  $k_2 \geq \max\{\frac{1-\delta}{M}, \frac{48k_1 M}{\delta^2}\}$ , then  $x_1, x_2 \in \mathcal{L}_\infty$ .

**Proof.** The closed-loop system is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \sigma(-k_1 x_1 - k_2 x_2) + d_1 + d_2. \end{aligned}$$

We construct a fictitious state

$$\begin{aligned} \dot{y} &= \sigma(-k_1x_1 - k_2x_2) - \sigma(-k_1x_1 - k_2x_2 + k_2y) + d_2, \\ y(0) &= 0. \end{aligned}$$

Lemma 3 shows that  $|y(t)| \leq 2M$  for all  $t \geq 0$ . Similar to the proof of Theorem 2, we define  $z = x_2 - y$  and convert the closed-loop system to the form

$$\begin{aligned} \dot{x}_1 &= z + y, \\ \dot{z} &= \sigma(-k_1x_1 - k_2z) + d_1, \end{aligned}$$

with  $z(0) = x_2(0)$  and  $y \in \mathcal{L}_\infty(2M)$ .

We also introduce another fictitious state

$$\dot{w} = \sigma(-k_1x_1 - k_2z) - \sigma(-k_1x_1 - k_2z + k_2w - d_1)$$

with  $w(0) = 0$ . Following the same argument as in the proof of Lemma 4, we can show that  $|w(t)| \leq \frac{1-\delta}{k_2} \leq M$  for all  $t \geq 0$ . Define  $\xi_1 = x_1$ ,  $\xi_2 = z - w = x_2 - y - w$ . Then (1) can be transformed into

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + w + y, \\ \dot{\xi}_2 &= \sigma(-k_1\xi_1 - k_2\xi_2 - d_1) + d_1, \end{aligned}$$

where  $\xi_1(0) = x_1(0)$ ,  $\xi_2(0) = x_2(0)$  and  $|w(t) + y(t)| \leq M + 2M = 3M$  for all  $t \geq 0$ . Since  $w(t)$  and  $y(t)$  are bounded, we know that  $x_1$  and  $x_2$  are bounded if  $\xi_1$  and  $\xi_2$  are bounded.

Define  $\tilde{\sigma}_{d_1}(s) = \sigma(s - d_1) + d_1$  with  $|d_1| \leq 1 - \delta$ . Then

$$\tilde{\sigma}_{d_1}(s) = \begin{cases} 1 + d_1, & s \geq 1 + d_1, \\ s, & -1 + d_1 \leq s < 1 + d_1, \\ -1 + d_1, & s \leq -1 + d_1. \end{cases} \quad (6)$$

This function can be viewed as a generalized saturation function, which is visualized in Fig. 2. It is easy to verify that

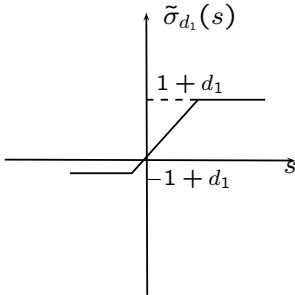


Fig. 2. Generalized saturation function  $\tilde{\sigma}_{d_1}(s)$

$\tilde{\sigma}_{d_1}$  satisfies the following properties:

- (1)  $|\tilde{\sigma}_{d_1}(s)| \leq 2$
- (2)  $s\tilde{\sigma}_{d_1}(s) \geq 0$  and  $s\tilde{\sigma}_{d_1}(s) = 0$  iff  $s = 0$

$$(3) \quad s [\tilde{\sigma}_{d_1}(v+s) - \tilde{\sigma}_{d_1}(v)] \geq 0$$

Moreover, it is shown in Lemma 9 in the Appendix that if  $|v| \leq \frac{\delta}{2}$ , then

$$s [\tilde{\sigma}_{d_1}(v+s) - \tilde{\sigma}_{d_1}(v)] \geq s\sigma_{\delta/2}(s),$$

where  $\sigma_{\delta/2}(s)$  is the standard saturation function with saturation level  $\delta/2$ , which is defined by  $\sigma_{\delta/2}(s) = \text{sign}(s) \min\{\delta/2, |s|\}$ .

With this generalized saturation function, the closed-loop system can be rewritten as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + w + y, \\ \dot{\xi}_2 &= \tilde{\sigma}_{d_1}(-k_1\xi_1 - k_2\xi_2). \end{aligned}$$

Define a positive definite function

$$V = \int_0^{k_1\xi_1} \tilde{\sigma}_{d_1}(s) ds + \int_0^{k_1\xi_1 + k_2\xi_2} \tilde{\sigma}_{d_1}(s) ds + k_1\xi_2^2.$$

Differentiating  $V$  along the trajectory yields

$$\begin{aligned} \dot{V} &= (k_1\xi_2 + k_1w + k_1y)\tilde{\sigma}_{d_1}(k_1\xi_1) - 2k_1\xi_2\tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2) + \\ &\quad [k_1\xi_2 + k_1w + k_1y - k_2\tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)] \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2) \\ &= k_1\xi_2 [\tilde{\sigma}_{d_1}(k_1\xi_1) - \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)] - k_2\tilde{\sigma}_{d_1}^2(k_1\xi_1 + k_2\xi_2) \\ &\quad + k_1(w+y) [\tilde{\sigma}_{d_1}(k_1\xi_1) + \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)] \\ &\leq k_1\xi_2 [\tilde{\sigma}_{d_1}(k_1\xi_1) - \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)] - k_2\tilde{\sigma}_{d_1}^2(k_1\xi_1 + k_2\xi_2) \\ &\quad + 12k_1M. \end{aligned}$$

If  $|k_1\xi_1 + k_2\xi_2| \geq \frac{\delta}{2}$ , then

$$-k_2\tilde{\sigma}_{d_1}^2(k_1\xi_1 + k_2\xi_2) + 12k_1M \leq -\frac{48k_1M}{\delta^2} \frac{\delta^2}{4} + 12k_1M = 0,$$

and hence  $\dot{V} \leq 0$ . If  $|k_1\xi_1 + k_2\xi_2| \leq \frac{\delta}{2}$  and  $|\xi_2| \geq \max\{\frac{\delta}{2k_2}, \frac{24M}{\delta}\}$ , then by using Lemma 9 we have

$$\begin{aligned} k_1\xi_2 [\tilde{\sigma}_{d_1}(k_1\xi_1) - \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)] &\leq \\ -k_1\xi_2\sigma_{\delta/2}(k_2\xi_2) &\leq -k_1\frac{24M}{\delta}\frac{\delta}{2} \leq -12k_1M, \end{aligned}$$

and hence  $\dot{V} \leq 0$ . We therefore find that  $\dot{V} \leq 0$  outside the region defined by  $|k_1\xi_1 + k_2\xi_2| \leq \frac{1}{2}$  and  $|\xi_2| \leq \max\{\frac{\delta}{2k_2}, \frac{24M}{\delta}\}$ . It follows that  $V$  remains bounded, which implies that  $\xi_1$  and  $\xi_2$  remain bounded. ■

Our final result concerns a special case where the disturbance consists of a finite number of sinusoids together with a DC bias of magnitude less than 1. In this case, any internally stabilizing linear static feedback controller guarantees that the states of the system (1) remain bounded.

**Theorem 7** Consider the system (1) with  $k_1 > 0$  and  $k_2 > 0$ . Suppose that  $d = d_1 + d_2$ , where  $d_1$  is a constant satisfying  $|d_1| < 1$  and  $d_2$  is generated by an exogenous system

$$\begin{aligned}\dot{w} &= Aw, \quad w(0) = w_0, \\ d &= Cw,\end{aligned}$$

where  $A$  is non-singular and satisfies  $A + A' = 0$ . We have  $x_1, x_2 \in \mathcal{L}_\infty$  for any initial condition.

**Proof.** We can rewrite the closed-loop system in a compact form:

$$\begin{bmatrix} \dot{w} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [\sigma(-k_1 x_1 - k_2 x_2) + d_1].$$

Consider the state transformation

$$\begin{bmatrix} w \\ \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -CA^{-2} & 1 & 0 \\ -CA^{-1} & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x_1 \\ x_2 \end{bmatrix}.$$

This transformation results in the system

$$\begin{bmatrix} \dot{w} \\ \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times [\sigma(-(k_1 CA^{-2} + k_2 CA^{-1})w - k_1 \bar{x}_1 - k_2 \bar{x}_2) + d_1].$$

Define  $v = -(k_1 CA^{-2} + k_2 CA^{-1})w + d_1$ . Then

$$\begin{aligned}\sigma(-(k_1 CA^{-2} + k_2 CA^{-1})w - k_1 \bar{x}_1 - k_2 \bar{x}_2) + d_1 \\ = \sigma(v - k_1 \bar{x}_1 - k_2 \bar{x}_2 - d_1) + d_1 \\ = \tilde{\sigma}_{d_1}(-k_1 \bar{x}_1 - k_2 \bar{x}_2 + v),\end{aligned}$$

where  $\tilde{\sigma}_{d_1}$  is the generalized saturation function defined in the proof of Theorem 6. The dynamics of  $\bar{x}_1$  and  $\bar{x}_2$  can now be written as

$$\begin{aligned}\dot{\bar{x}}_1 &= \bar{x}_2, \\ \dot{\bar{x}}_2 &= \tilde{\sigma}_{d_1}(-k_1 \bar{x}_1 - k_2 \bar{x}_2 + v).\end{aligned}$$

Clearly  $v \in \mathcal{L}_\infty$ . It was shown by Chitour (2001) that the  $(\bar{x}_1, \bar{x}_2)$  dynamics is  $\mathcal{L}_\infty$  stable from  $v$  to  $\bar{x}_1$  and  $\bar{x}_2$  for any  $k_1 > 0$  and  $k_2 > 0$ . ■

**Remark 8** For ease of presentation, we use a standard saturation function with saturation level 1, but all the results obtained in this paper can easily be extended to the case where a saturation function with arbitrary saturation level  $\Delta$  is used.

## Appendix

*Intersection problem in the proof of Theorem 1*

We shall show that

$$y_1(\tilde{t}) = -\frac{\lambda}{2}\tilde{t}^2 + (N\lambda + \lambda)\tilde{t} + 2\lambda(1 - \cos(\pi\tilde{t})) - \frac{2\lambda}{\pi}\sin(\pi\tilde{t}) + 1$$

has only one intersection with  $y_1 = 1$  for  $\tilde{t} > 0$  and sufficiently large  $N$ . Let

$$\tilde{t}_1 = \min\{\tilde{t} > 0 : y_1(\tilde{t}) = 1\}, \quad \tilde{t}_0 = \min\left\{\tilde{t} > 0 : \frac{dy_1}{d\tilde{t}}(\tilde{t}) = 0\right\}.$$

Given  $y_1(0) = 1$  and  $\frac{dy_1}{d\tilde{t}} > 0$ , we must have  $\tilde{t}_1 > \tilde{t}_0$ . Note that

$$\frac{dy_1}{d\tilde{t}}(\tilde{t}) = -\lambda\tilde{t} + N\lambda + \lambda + 2\lambda\pi\sin(\pi\tilde{t}) - 2\lambda\cos(\pi\tilde{t}).$$

Hence  $\tilde{t}_0 \geq N + 1 - 2\pi - 2 > \frac{N}{2}$  for sufficiently large  $N$ . However,

$$y_1(\tilde{t}) \geq -\frac{\lambda}{2}\tilde{t}^2 + (N\lambda + \lambda)\tilde{t} - 4\lambda - 2\lambda + 1$$

Hence we have

$$-\frac{\lambda}{2}\tilde{t}_1^2 + (N\lambda + \lambda)\tilde{t}_1 - 4\lambda - 2\lambda \leq 0$$

or equivalently

$$\frac{1}{2}\tilde{t}_1^2 - (N + 1)\tilde{t}_1 + 6 \geq 0$$

This implies

$$\tilde{t}_1 < r_1, \text{ or } \tilde{t}_1 > r_2$$

where  $r_1$  and  $r_2$  are two roots of  $\frac{1}{2}\tilde{t}^2 - (N + 1)\tilde{t} + 6$ .

$$r_{1,2} = N + 1 \mp \sqrt{(N + 1)^2 - 12}$$

Note that

$$\begin{aligned}r_1 &= N + 1 - \sqrt{(N + 1)^2 - 12} \\ &= \frac{12}{N + 1 + \sqrt{(N + 1)^2 - 12}} \leq \frac{N}{2}\end{aligned}$$

for sufficiently large  $N$ . Since we already know  $\tilde{t}_1 > \tilde{t}_0 > \frac{N}{2}$ , we must have  $\tilde{t}_1 > r_2$ . But then

$$\begin{aligned}r_2 &= N + 1 + \sqrt{(N + 1)^2 - 12} \geq N + 1 + \sqrt{(N + 1)^2 / 4} \\ &= \frac{3}{2}(N + 1)\end{aligned}$$

for large  $N$ . We find that  $\tilde{t}_1 > \frac{3}{2}(N+1)$ . But for  $\tilde{t} > \frac{3}{2}(N+1)$ , we have

$$\begin{aligned} \frac{dy_1}{d\tilde{t}}(\tilde{t}) &< -\frac{3}{2}\lambda(N+1) + N\lambda + \lambda + 2\lambda\pi \sin(\pi\tilde{t}) - 2\lambda \cos(\pi\tilde{t}) \\ &< -\frac{1}{2}\lambda(N+1) + 2\lambda\pi \sin(\pi\tilde{t}) - 2\lambda \cos(\pi\tilde{t}) < 0 \end{aligned}$$

for sufficiently large  $N$ . This shows that  $y_1(\tilde{t}) < 1$  for all  $\tilde{t} > \tilde{t}_1$ , and hence, the only intersection with  $y_1 = 1$  is at  $\tilde{t} = \tilde{t}_1$ .

*Property of  $\tilde{\sigma}_{d_1}(s)$*

**Lemma 9** *The generalized saturation function  $\tilde{\sigma}_{d_1}$  defined in (6) with  $|d_1| \leq 1 - \delta$  satisfies*

$$s [\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v)] \geq s\sigma_{\delta/2}(s)$$

for  $|v| \leq \frac{\delta}{2}$  where  $\sigma_{\delta/2}(s)$  denotes the standard saturation function with saturation level  $\delta/2$  defined as  $\sigma_{\delta/2}(s) = \text{sign}(s) \min\{\delta/2, |s|\}$ .

**Proof.** If  $|s| < \frac{\delta}{2}$ , we have  $|v+s| \leq \delta \leq 1 - |d_1|$ . By definition (6), we have  $\tilde{\sigma}_{d_1}(s+v) = s+v$ . Hence

$$\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v) = s+v-v = s.$$

If  $|s| \geq \frac{\delta}{2}$ , it can be seen from Fig. 2 that

$$|\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v)| \geq |\text{sign}(s)\frac{\delta}{2} + v - v| = \frac{\delta}{2}.$$

Hence  $s [\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v)] \geq s\sigma_{\delta/2}(s)$ . ■

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