Stabilization of a class of sandwich systems via state feedback

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***Abstract*—We consider the problem of state-feedback stabilization for a class of sandwich systems, consisting of two linear systems connected in cascade via a saturation. In particular, we present design methodologies for constructing semiglobally and globally stabilizing controllers for such systems when the input is itself subject to saturation. The design is carried out under a set of assumptions that are proven to be both necessary and sufficient. The presented design methodologies are extensions of classical low-gain design methodologies developed for stabilizing linear systems subject to input saturation. The methodologies can be further extended to multilayer sandwich systems, consisting of an arbitrary number of cascaded linear systems with saturations sandwiched between them.**

I. Introduction

Many physical systems can be modeled as interconnections of several distinct subsystems, some of which are linear and some of which are nonlinear. One common type of structure consists of two linear systems connected in cascade via a static nonlinearity. We refer to such systems as *sandwich systems*, because the static nonlinearity is *sandwiched* between the two linear systems.

In this paper we focus on sandwich systems where the sandwiched nonlinearity is a saturation. Saturations can occur due to the limited capacity of an actuator, limited range of a sensor, or physical limitations within a system. Physical quantities such as speed, acceleration, pressure, flow, current, voltage, and so on, are always limited to a finite range, and saturations are therefore a ubiquitous feature of physical systems. Our primary goal is to develop design methodologies for semiglobal and global stabilization of such systems by state feedback. To make our design more general, we also assume that the input is subject to saturation. The resulting system configuration is illustrated in Fig. 1.

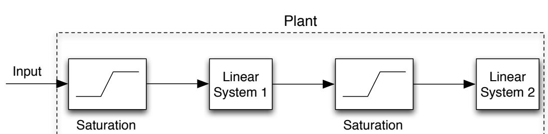

Fig. 1. Sandwich system subject to input saturation

In the absence of an input saturation, sandwich systems are a special case of *cascade systems*, where the output of a linear system affects a nonlinear system. Studies on such systems was initiated in [1] and continued elsewhere, for example, in [2]. In [1], [2] the nonlinear system is assumed to be stable, and the goal is to investigate whether instability can occur when the linear system is also stable. By contrast, the goal of this paper is construction of stabilizing controllers for the overall sandwich system.

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Stabilization of sandwich systems has been studied previously, for example, by Taware and Tao (see [3]). The main technique used in [3], and in other related works, is based on approximate inversion of the sandwiched nonlinearity. Inversion is a viable approach for some types of nonlinearities, a prominent example being the deadzone nonlinearity, which is right-invertible. Saturations, however, have a limited range and are therefore not amenable to inversion except in a small region; thus, a different approach is required.

The problem considered in this paper is related to the problem of stabilizing a single linear system subject to input saturation. Several important results on this topic have appeared in the literature, starting with the works of Fuller [4], [5] and continuing with the works of Sontag, Sussmann, and Yang [6]–[8] (see also [9], [10]). These works led to the development of *low-gain* design methodologies for semiglobal stabilization, and *scheduled low-gain* design methodologies for global stabilization [11], [12]. The scheduled low-gain design methodology is based on the concept of scheduling, developed by Megretski [13]. Also, in the context of global stabilization, another design methodology that was introduced is the nested saturation methodology [14].

Recent research has also focused on linear systems subject to state constraints, where the controller must guarantee that the output of a linear system remains in a given set (see, e.g., [15]). Such an approach can be used to control sandwich systems, by designing controllers in order to avoid saturation altogether. However, this is only possible for initial conditions belonging to some bounded set of admissible initial conditions, and the approach can therefore not be used for semiglobal or global stabilization.

The design methodologies presented in this paper are generalizations of the classical low-gain and scheduled low-gain design methodologies for stabilization of linear systems subject to input saturation, and we therefore refer to them as *generalized* low-gain design methodologies. We also discuss how these methodologies can be extended to handle *multilayer sandwich systems*, consisting of an arbitrary number of cascaded linear systems with saturations sandwiched between them.

II. Problem formulation

We consider the sandwich system illustrated in Fig. 1, described by the following equations:

$$L_1 : \begin{cases} \dot{x} = Ax + B\sigma(u), & x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, \\ z = Cx, & z \in \mathbb{R}^q, \end{cases} \tag{1a}$$

$$L_2 : \quad \dot{\omega} = M\omega + N\sigma(z), \quad \omega \in \mathbb{R}^m. \tag{1b}$$

The function $\sigma(\cdot)$ represents a standard component-wise saturation with limits at ± 1.

To simplify the exposition, we define the state vector $\chi = [x', \omega']'$, which combines the states of the L_1 and L_2 subsystems. When both of the saturations in (1) are inactive, the dynamics of the system are described by the linear system equations

$$\dot{\chi} = \mathscr{A}\chi + \mathscr{B}u, \quad \mathscr{A} = \begin{bmatrix} A & 0 \\ NC & M \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{2}$$

Our goal is to design state-feedback controllers to stabilize the system (1), and toward this end, we make the following assumption:

Assumption 1: The pair $(\mathscr{A}, \mathscr{B})$ is stabilizable and the eigenvalues of $\mathscr{A}$ are located in the closed left-half plane.

Remark 1: Note that, due to the cascaded structure of the system, the eigenvalues of $\mathscr{A}$ consist of the eigenvalues of A together with the eigenvalues of M.

III. Necessary and sufficient conditions for stabilizability

We say that the origin of the system (1) is *semiglobally stabilizable* if, for each compact set $\mathscr{W} \subset \mathbb{R}^{n+m}$, there exists a state-feedback controller that renders the origin asymptotically stable with $\mathscr{W}$ contained in the region of attraction. We say that the origin is *globally stabilizable* if there exists a state-feedback controller that renders the origin globally asymptotically stable. The following theorem relates these notions of stabilizability to the conditions in Assumption 1:

Theorem 1: The origin of (1) is semiglobally stabilizable if, and only if, Assumption 1 is satisfied. Similarly, the origin is globally stabilizable if, and only if, Assumption 1 is satisfied.

Proof: Necessity of the conditions in Assumption 1 is established by noting that the system (1) can only be semiglobally or globally stabilizable if the linear system description (2), which is valid locally around the origin, is stabilizable. Hence, the pair $(\mathscr{A},\mathscr{B})$ must be stabilizable. Furthermore, it is known from [16] that a linear time-invariant system can only be semiglobally or globally stabilized by a saturated input if the eigenvalues of the system are confined to the closed left-half plane. Both the L_1 and L_2 subsystems must be stabilized through saturated inputs, and hence the eigenvalues of A and M (and therefore of $\mathscr{A}$) must be in the closed left-half plane. Sufficiency is established by constructive design of stabilizing controllers in Section IV. ■

As Theorem 1 shows, the conditions for semiglobal and global stabilizability are the same. The intrinsic difference between the two cases lies in the type of controller that can be used: semiglobal stabilization can be achieved with a linear controller, whereas global stabilization can in general only be achieved with a nonlinear controller (see [4]).

IV. Generalized low-gain design

The design methodologies presented in this paper are generalizations of classical low-gain design methodologies for linear systems subject to input saturation. The principle behind classical low-gain design is to create a control law with a sufficiently low gain to keep the input saturation inactive for all time. In the semiglobal case, the gain is fixed, based on an *a priori* given set of admissible initial conditions; in the global case, the gain is scheduled to be sufficiently low regardless of the initial conditions.

For the systems considered in this paper, the principle is similar. However, there are now two saturations, and the problem is more complex because the sandwiched saturation cannot be made inactive from the start by using low gain. Instead, the sandwiched saturation must be deactivated by controlling the states of the L_1 subsystem toward the origin. Conceptually, the control task can therefore be viewed as consisting of two subtasks. The first subtask is to control the states of the L_1 subsystem toward the origin, in order to deactivate the sandwiched saturation. Once the sandwiched saturation has been deactivated, the second subtask consists of controlling the state of the whole system to the origin without reactivating the sandwiched saturation. All of this should be accomplished without activating the input saturation.

To accomplish the two subtasks, the control law is divided into two terms, referred to as the L_1 *term* and the L_1/L_2 *term*. The L_1 term is a function of x, and the purpose of this term is to control the state of the L_1 subsystem toward the origin, in order to permanently deactivate the sandwiched saturation. The gain used in this term is chosen sufficiently low to avoid activating the input saturation, by adjusting a low-gain parameter $\varepsilon_1 > 0$. The L_1/L_2 term is a function of x and ω, and the purpose of this term is to control the states of both subsystems to the origin once the sandwiched saturation becomes inactive. The gain of this term is chosen sufficiently low that it does not interfere with the L_1 term's ability to permanently deactivate the sandwiched saturation, by adjusting a low-gain parameter $\varepsilon_2 > 0$.

A. Semiglobal stabilization

To construct a semiglobally stabilizing class of controllers, we begin by letting P_{ε_1} denote the unique symmetric positive-definite solution of the algebraic Riccati equation (ARE)

$$A'P_{\varepsilon_1} + P_{\varepsilon_1}A - P_{\varepsilon_1}BB'P_{\varepsilon_1} + \varepsilon_1 I_n = 0. \tag{3}$$

Define $F_{\varepsilon_1} := -B'P_{\varepsilon_1}$ and $\bar{F} := [F_{\varepsilon_1}, 0] \in \mathbb{R}^{p\times(n+m)}$. We continue by letting $\mathscr{P}_{\varepsilon_2}$ denote the unique symmetric positive-definite solution of the ARE

$$(\mathscr{A}+\mathscr{B}\bar{F})'\mathscr{P}_{\varepsilon_2} + \mathscr{P}_{\varepsilon_2}(\mathscr{A}+\mathscr{B}\bar{F}) - \mathscr{P}_{\varepsilon_2}\mathscr{B}\mathscr{B}'\mathscr{P}_{\varepsilon_2} + \varepsilon_2 I_{n+m} = 0. \tag{4}$$

Define $\mathscr{F}_{\varepsilon_2} := -\mathscr{B}'\mathscr{P}_{\varepsilon_2}$. The system (1) is now semiglobally stabilized by the control law

$$u = F_{\varepsilon_1}x + \mathscr{F}_{\varepsilon_2}\chi. \tag{5}$$

In terms of our previous discussion, the term $F_{\varepsilon_1}x$ is the L_1 term and the term $\mathscr{F}_{\varepsilon_2}\chi$ is the L_1/L_2 term. The low-gain parameters $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ must be chosen sufficiently small depending on the size of the set of admissible initial conditions, as shown by the following theorem:

Theorem 2: Let $\mathscr{W} \subset \mathbb{R}^{n+m}$ be a compact set, and suppose that Assumption 1 is satisfied. Then there exists an $\varepsilon_1^* > 0$ such that for each $0 < \varepsilon_1 < \varepsilon_1^*$, there exists an $\varepsilon_2^*(\varepsilon_1) > 0$ such that for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$, the controller described by (5) renders the origin of (1) asymptotically stable with $\mathscr{W}$ contained in the region of attraction.

Proof: Consider first the system description (2) with $u = F_{\varepsilon_1}x + \mathscr{F}_{\varepsilon_2}\chi$, which is valid locally around the origin where both saturations are inactive. Defining the Lyapunov function candidate $V(\chi) = x'P_{\varepsilon_1}x + \chi'\mathscr{P}_{\varepsilon_2}\chi$, it is easily confirmed that we obtain the time derivative

$$\begin{aligned}\dot{V}(\chi) &= -\varepsilon_1 x'x - x'P_{\varepsilon_1}BB'P_{\varepsilon_1}x - 2x'P_{\varepsilon_1}B\mathscr{B}'\mathscr{P}_{\varepsilon_2}\chi - \varepsilon_2\chi'\chi \\ &\quad - \chi'\mathscr{P}_{\varepsilon_2}\mathscr{B}\mathscr{B}'\mathscr{P}_{\varepsilon_2}\chi \\ &= -\varepsilon_1 x'x - \varepsilon_2\chi'\chi - (B'P_{\varepsilon_1}x + \mathscr{B}'\mathscr{P}_{\varepsilon_2}\chi)'(B'P_{\varepsilon_1}x + \mathscr{B}'\mathscr{P}_{\varepsilon_2}\chi).\end{aligned}$$

Thus, we know that the system is locally exponentially stable. Since $\chi(0)$ belongs to the compact set $\mathscr{W}$, there exist compact sets X and Ω such that $x(0) \in X$ and $\omega(0) \in \Omega$.

Because the eigenvalues of $\mathscr{A}$, and therefore the eigenvalues of A, are in the closed left-half plane, the solutions of (3) are such that $P_{\varepsilon_1} \to 0$ as $\varepsilon_1 \to 0$ [12, Lemma 2.2.6]. Furthermore, the matrix $F_{\varepsilon_1} = -B'P_{\varepsilon_1}$ is such that the matrix $A+BF_{\varepsilon_1}$ is Hurwitz, and it follows that the eigenvalues of the matrix $\mathscr{A}+\mathscr{B}\bar{F}$ are in the closed left-half plane. This in turn implies that for each $\varepsilon_1 > 0$, the solutions of (4) are such that $\mathscr{P}_{\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$. From these considerations, we may conclude that $\lim_{\varepsilon_1\to 0} F_{\varepsilon_1} = 0$, and for each $\varepsilon_1 > 0$, $\lim_{\varepsilon_2 \to 0}\mathscr{F}_{\varepsilon_2} = 0$.

We first investigate the effect of the L_1 term alone; that is, the feedback matrix F_{ε_1}. Since the matrix $A+BF_{\varepsilon_1}$ is Hurwitz and $F_{\varepsilon_1} \to 0$ as $\varepsilon_1 \to 0$, there exists an $\varepsilon_1^* > 0$ such that for all $0 < \varepsilon_1 < \varepsilon_1^*$ and for all $x(0) \in X$, the input saturation remains inactive in the sense that $\|F_{\varepsilon_1}x(t)\| = \|F_{\varepsilon_1}\mathrm{e}^{(A+BF_{\varepsilon_1})t}x(0)\| \le \frac{1}{4}$ (see [17, Theorem 2.8]). Let ε_1 be fixed such that this inequality is satisfied, and define $\gamma > 0$ such that $x'P_{\varepsilon_1}x \le \gamma$ implies $\|Cx\| \le \frac{1}{4}$ and $\|F_{\varepsilon_1}x\| \le \frac{1}{4}$. Define $K = \{x \in \mathbb{R}^n \mid x'P_{\varepsilon_1}x \le \gamma\}$, and let $T > 0$ be chosen large enough that for all $x(0) \in X$, $x(T) = \mathrm{e}^{(A+BF_{\varepsilon_1})T}x(0) \in K$.

Next, consider the complete control law, with both the L_1 and the L_1/L_2 terms; that is, $u = F_{\varepsilon_1}x + \mathscr{F}_{\varepsilon_2}\chi$. The L_1/L_2 term can be partitioned as $\mathscr{F}_{\varepsilon_2}\chi = \mathscr{F}_{1,\varepsilon_2}x + \mathscr{F}_{2,\varepsilon_2}\omega$, where $\mathscr{F}_{1,\varepsilon_2} \to 0$ and $\mathscr{F}_{2,\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$. Since $\omega(0) \in \Omega$ and the input $\sigma(z)$ to the L_2 subsystem is bounded, we know that there exists a compact set $\bar{\Omega} \supset \Omega$ such that for all $t \in [0,T]$, $\omega(t) \in \bar{\Omega}$. Using the property that $\mathscr{F}_{2,\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$, we therefore see that the term $\mathscr{F}_{2,\varepsilon_2}\omega$ can be made arbitrarily small on the time interval $[0,T]$ by decreasing ε_2. This, combined with the property that $\mathscr{F}_{1,\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$, shows that for small ε_2, the control law on the interval $[0,T]$ can be viewed as a small perturbation of the control law $u = F_{\varepsilon_1}x$. Thus, we know that for all sufficiently small ε_2, $x(T) \in 2K$ is satisfied for all $\chi(0) \in \mathscr{W}$. Accordingly, let $\varepsilon_2^*(\varepsilon_1)$ be chosen small enough that, for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$ and all $\chi(0) \in \mathscr{W}$, we have $x(T) \in 2K$. Furthermore, let $\varepsilon_2^*(\varepsilon_1)$ be chosen small enough that the following two properties hold for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$: (i) $x'P_{\varepsilon_1}x \le 4\gamma$ and $\omega \in \bar{\Omega}$ implies $V(\chi) \le 9\gamma$; and (ii) $V(\chi) \le 9\gamma$ implies $\|\mathscr{F}_{\varepsilon_2}\chi\| \le \frac{1}{4}$.

We can now make several observations. At time T, we know that $x(T) \in 2K$ and $\omega(T) \in \bar{\Omega}$, which means that $x'(T)P_{\varepsilon_1}x(T) \le 4\gamma$, and thus we can conclude that $V(\chi(T)) \le 9\gamma$. Furthermore, for all χ such that $V(\chi) \le 9\gamma$, we have $x'P_{\varepsilon_1}x \le 9\gamma$, which means that $x \in 3K$. This in turn implies that $\|F_{\varepsilon_1}x\| \le \frac{3}{4}$ and $\|Cx\| \le \frac{3}{4}$. Combined with the expression $\|\mathscr{F}_{\varepsilon_2}\chi\| \le \frac{1}{4}$, this implies that $\|u\| = \|F_{\varepsilon_1}x + \mathscr{F}_{\varepsilon_2}\chi\| \le 1$. Thus, for all χ such that $V(\chi) \le 9\gamma$, both the input saturation and the sandwiched saturation are inactive. The proof is completed by noting that when both saturations are inactive, $V(\chi)$ is a Lyapunov function. Thus, χ never escapes from the level set defined by $V(\chi) \le 9\gamma$, and the system therefore behaves like a linear, exponentially stable system for all $t \ge T$. ∎

Remark 2: To implement the semiglobally stabilizing controller, it is necessary to find appropriate low-gain parameters ε_1 and ε_2. It is difficult to derive tight upper bounds ε_1^* and $\varepsilon_2^*(\varepsilon_1)$ analytically, and thus the parameters are typically found experimentally, by gradually decreasing them until the desired stability is achieved.

B. Global stabilization

To achieve global stabilization, we use a control law that is very similar to the semiglobal case. The main difference is that, instead of being fixed, the low-gain parameters are scheduled as functions of the state of the system.

Let $P_{\varepsilon_1(x)}$ be the unique symmetric positive-definite solution of the ARE (3) with $\varepsilon_1 = \varepsilon_1(x)$. Define $F_{\varepsilon_1(x)} := -B'P_{\varepsilon_1(x)}$ and $\bar{F} := [F_1, 0] \in \mathbb{R}^{p\times(n+m)}$ (where $F_1 = -B'P_1$ and P_1 is the solution of (3) with $\varepsilon_1 = 1$). Let $\mathscr{P}_{\varepsilon_2(\chi)}$ be the unique symmetric positive-definite solution of the ARE (4) with $\varepsilon_2 = \varepsilon_2(\chi)$. Define $\mathscr{F}_{\varepsilon_2(\chi)} = -\mathscr{B}'\mathscr{P}_{\varepsilon_2(\chi)}$. When the scheduled low-gain parameters $\varepsilon_1(x)$ and $\varepsilon_2(\chi)$ are properly defined, the system (1) is globally stabilized by the control law

$$u = F_{\varepsilon_1(x)}x + \varepsilon_1(x)\mathscr{F}_{\varepsilon_2(\chi)}\chi. \tag{6}$$

In terms of our previous discussion, the term $F_{\varepsilon_1(x)}x$ is the L_1 term and the term $\varepsilon_1(x)\mathscr{F}_{\varepsilon_2(\chi)}\chi$ is the L_1/L_2 term.

We now specify our requirements for the scheduled low-gain parameters $\varepsilon_1(x)$ and $\varepsilon_2(\chi)$. The function $\varepsilon_1 \colon \mathbb{R}^n \to (0,1]$ must be continuous and satisfy the following properties:

1) There exists an open neighborhood O of the origin such that for all $x \in O$, $\varepsilon_1(x) = 1$.
2) For any $x \in \mathbb{R}^n$, $\|B'P_{\varepsilon_1(x)}x\| \le \frac{1}{2}$.
3) $\varepsilon_1(x) \to 0 \implies \|x\| \to \infty$.
4) For each $c > 0$, the set $\{x \in \mathbb{R}^n \mid x'P_{\varepsilon_1(x)}x \le c\}$ is bounded.
5) There is a function $g\colon \mathbb{R}_{>0} \to (0,1]$ such that for all $x \ne 0$, $\varepsilon_1(x) = g(x'P_{\varepsilon_1(x)}x)$.

A particular choice that satisfies the above conditions is

$$\varepsilon_1(x) = \max\left\{ r \in (0,1] \mid x'P_r x \cdot \mathrm{trace}(B'P_rB) \le \frac{1}{4} \right\} \tag{7}$$

(where P_r is the solution of (3) with $\varepsilon_1 = r$).

To define $\varepsilon_2(\chi)$, first define

$$\begin{aligned} &\delta := \min\left\{ \frac{1}{2}, \frac{\ell}{4\|F_1\|}, \frac{1}{2\rho} \right\}, \quad \ell := \frac{1}{2\sqrt{\|P_1\|\mathrm{trace}(B'P_1B)}}, \\ &\rho := \int_0^\infty \|C\mathrm{e}^{(A+BF_1)t}B\|\,\mathrm{d}t. \end{aligned} \tag{8}$$

Note that ρ is well-defined because $A+BF_1$ is Hurwitz. The function $\varepsilon_2\colon \mathbb{R}^{n+m} \to (0,1]$ must be continuous and satisfy properties 1–4 above, with x replaced by χ, B replaced by $\mathscr{B}$, $P_{\varepsilon_1(x)}$ replaced by $\mathscr{P}_{\varepsilon_2(\chi)}$, and the number $\frac{1}{2}$ in Property 2 replaced by δ. A particular choice that satisfies these conditions is

$$\varepsilon_2(\chi) = \max\left\{ r \in (0,1] \mid \chi'\mathscr{P}_r\chi \cdot \mathrm{trace}(\mathscr{B}'\mathscr{P}_r\mathscr{B}) \le \delta^2 \right\} \tag{9}$$

(where $\mathscr{P}_r$ is the solution of (4) with $\varepsilon_2 = r$).

Theorem 3: Suppose that Assumption 1 is satisfied. Then the controller described by (6), with $\varepsilon_1(x)$ and $\varepsilon_2(\chi)$ defined by (7), (9), renders the origin of (1) globally asymptotically stable.

Proof: We start by noting that the properties of the scheduling guarantee that $\|F_{\varepsilon_1(x)}x\| = \|B'P_{\varepsilon_1(x)}x\| \le \frac{1}{2}$ and $\|\varepsilon_1(x)\mathscr{F}_{\varepsilon_2(\chi)}\chi\| \le \|\mathscr{B}'\mathscr{P}_{\varepsilon_2(\chi)}\chi\| \le \delta \le \frac{1}{2}$. It follows that $\|u\| \le 1$, and hence the input saturation is always inactive.

For sufficiently small χ, both saturations are inactive, and we have $\varepsilon_1(x) = \varepsilon_2(\chi) = 1$. Thus, the system behaves like a linear system with a linear control law $u = F_1x + \mathscr{F}_1\chi$ in a region around the origin. As in the semiglobal case, it is easy to show that the origin of the resulting system is locally exponentially stable by using the Lyapunov function $V(\chi) = x'P_1x + \chi'\mathscr{P}_1\chi$.

Define $K = \{x \in \mathbb{R}^n \mid \varepsilon_1(x) = 1\}$. We wish to show that whenever $x \notin K$, $\varepsilon_1(x)$ is strictly increasing with respect to time. Suppose, for the sake of establishing a contradiction, that $\varepsilon_1(x)$ is not strictly increasing when $x \notin K$, that is, $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) \le 0$. Then we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x'P_{\varepsilon_1(x)}x) &= -\varepsilon_1(x)x'x - x'P_{\varepsilon_1(x)}BB'P_{\varepsilon_1(x)}x \\ &\quad - 2\varepsilon_1(x)x'P_{\varepsilon_1(x)}B\mathscr{B}'\mathscr{P}_{\varepsilon_2(\chi)}\chi + x'\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\varepsilon_1(x)}\right)x. \end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) \le 0$, the properties of the ARE imply that $\frac{\mathrm{d}}{\mathrm{d}t}P_{\varepsilon_1(x)} \le 0$. Furthermore,

$$\begin{aligned} \|2\varepsilon_1(x)x'P_{\varepsilon_1(x)}B\mathscr{B}'\mathscr{P}_{\varepsilon_2(\chi)}\chi\| &\le 2\varepsilon_1(x)\|x\|\|F_{\varepsilon_1(x)}\|\delta \\ &< \frac{2\delta}{\ell}\varepsilon_1(x)\|x\|^2\|F_1\| \le \frac{1}{2}\varepsilon_1(x)x'x, \end{aligned}$$

where we have used the properties $\|\mathscr{B}'\mathscr{P}_{\varepsilon_2(\chi)}\chi\| \le \delta \le \frac{\ell}{4\|F_1\|}$, $\|P_{\varepsilon_1(x)}B\| = \|F_{\varepsilon_1(x)}\| \le \|F_1\|$, and $x \notin K \implies \varepsilon_1(x) < 1 \implies \|x\| > \ell$. (The latter implication can be confirmed from (7) by noting that $\|x\| \le \ell \implies x'P_1x \cdot \mathrm{trace}(B'P_1B) \le \frac{1}{4}$.)

Combining the above expressions, we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(x'P_{\varepsilon_1(x)}x) \le -\frac{1}{2}\varepsilon_1(x)x'x < 0$. However, the properties of the scheduling then imply that $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) > 0$, which yields a contradiction with the assumption $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) \le 0$. We have therefore shown that $\varepsilon_1(x)$ is strictly increasing when $x \notin K$, which implies that x converges to, and remains in, K.

Let $t^* > 0$ be such that for all $t \ge t^*$, $x \in K$. Then for all $t \ge t^*$, $u = F_1x + v$, where $v = -\mathscr{B}\mathscr{P}_{\varepsilon_2(\chi)}\chi$. For all $t \ge t^*$, the output z of the L_1 subsystem is therefore described by

$$z(t) = C\mathrm{e}^{(A+BF_1)(t-t^*)}x(t^*) + \int_{t^*}^{t} C\mathrm{e}^{(A+BF_1)(t-\tau)}Bv(\tau)\,\mathrm{d}\tau.$$

The properties of the scheduling guarantee that $\|v\| \leq \delta \leq \frac{1}{2\rho}$. Let $T \geq t^*$ be such that for all $t \geq T$, $\|C\mathrm{e}^{(A+BF_1)(t-t^*)}x(t^*)\| \leq \frac{1}{2}$. Then for all $t \geq T$,

$$\begin{aligned}\|z(t)\| &\leq \|C\mathrm{e}^{(A+BF_1)(t-t^*)}x(t^*)\| + \left\|\int_{t^*}^{t} C\mathrm{e}^{(A+BF_1)(t-\tau)}Bv(\tau)\,\mathrm{d}\tau\right\| \\ &\leq \frac{1}{2} + \int_{t^*}^{t}\left\|C\mathrm{e}^{(A+BF_1)(t-\tau)}B\right\|\|v(\tau)\|\,\mathrm{d}\tau \\ &\leq \frac{1}{2} + \int_{0}^{\infty}\left\|C\mathrm{e}^{(A+BF_1)t}B\right\|\mathrm{d}\tau\,\frac{1}{2\rho} = 1.\end{aligned}$$

Hence, for all $t \geq T$, the sandwiched saturation is inactive, and the system is therefore described by the equation $\dot{\chi} = (\mathscr{A} + \mathscr{B}\bar{F})\chi - \mathscr{B}\mathscr{P}_{\varepsilon_2(\chi)}\chi$. From [13] we know that the origin of this system is globally asymptotically stable. ∎

Remark 3: To implement the globally stabilizing controller, one needs to calculate the parameter δ, which is used in the scheduling (9). This, in turn, requires calculating P_1, F_1, and ρ. P_1 is found by solving (3) with $\varepsilon_1 = 1$, and $F_1 = -B'P_1$. After F_1 has been found, ρ can be calculated by numerical integration according to (8).

C. Systems without input saturation

If the system is not subject to any input saturation, then the design task is simplified. In particular, there is no need to design the L_1 term using a low-gain strategy. The L_1 term can instead be designed simply as Fx, where F is any matrix such that $A+BF$ is Hurwitz. The design of the L_1/L_2 term can then be carried out as before with $\bar{F} = [F, 0] \in \mathbb{R}^{p\times(n+m)}$ (in the global case, by setting $\varepsilon_1(x) := 1$ where this variable appears in the L_1/L_2 term). The necessary and sufficient conditions for semiglobal and global stabilizability are also relaxed when no input saturation is present; in particular, only the eigenvalues of M are required to be in the closed left-half plane.

D. Multilayer sandwich systems

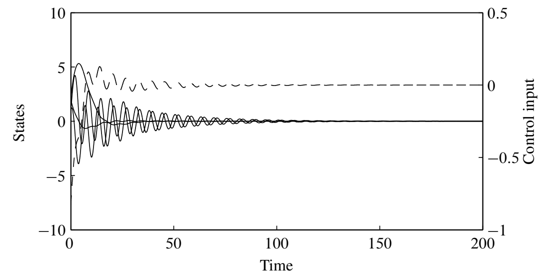
The generalized low-gain design methodology presented above can be extended to handle multilayer sandwich systems, consisting of an arbitrary number of cascaded linear systems with saturations sandwiched between them, with or without an additional input saturation. Consider, for example, a multilayer sandwich system consisting of three linear systems (L_1, L_2, and L_3), with two sandwiched saturations and an input saturation. Following the same approach as above, the control law for this system is divided into an L_1 term, an L_1/L_2 term, and an $L_1/L_2/L_3$ term. These terms are designed sequentially with low gains, to first ensure that the sandwiched saturation between the L_1 and L_2 subsystems is deactivated, then to ensure that the sandwiched saturation between the L_2 and L_3 subsystems is deactivated, and then to ensure that the states of all three subsystems are brought to the origin.

When there is no input saturation, necessary and sufficient conditions for semiglobal and global stabilization of multilayer sandwich systems are that (i) the local linear system is stabilizable; and (ii) the eigenvalues of the subsystems $L_2, L_3, \ldots$ are in the closed left-half plane. When the input is subject to saturation, the eigenvalues of the L_1 system must also be in the closed left-half plane.

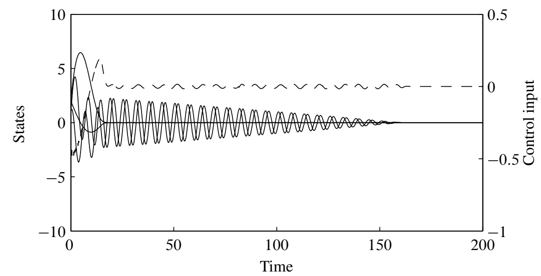
V. Simulation example

Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$



(a) States (solid, left axis) and control input (dashed, right axis) for semiglobally stabilizing controller



(b) States (solid, left axis) and control input (dashed, right axis) for globally stabilizing controller

Fig. 2. Simulation results

The L_1 subsystem has an eigenvalue at the origin of multiplicity two; thus, it is open-loop unstable. The L_2 subsystem has imaginary eigenvalues at $\pm 1j$; thus, it is marginally stable. Following the procedure in Section IV-A, we design a semiglobally stabilizing controller for this system with $\varepsilon_1 = 10^{-4}$ and $\varepsilon_2 = 5\cdot 10^{-4}$. Similarly, we design a globally stabilizing controller according the procedure in Section IV-B, which gives $\delta \approx 0.03$. Fig. 2 shows the simulation results with initial conditions $x(0) = [2,2]'$ and $\omega(0) = [1,1]'$.

VI. Concluding remarks

In this paper we have presented generalized low-gain design methodologies for semiglobal and global stabilization of sandwich systems subject to input saturation. We have chosen to focus on this particular type of system in order to best illustrate the principle of generalized low-gain design. As discussed in Sections IV-C and IV-D, however, the design methodology can be applied to a larger class of sandwich systems with saturations. Current research is focused on semiglobal and global stabilization by output feedback, as well as external stabilization problems.

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