Semi-global regulation of output synchronization for heterogeneous networks of non-introspective, invertible agents subject to actuator saturation

Tao Yang[1] Anton A. Stoorvogel[2] Håvard Fjær Grip[3] Ali Saberi[3]

Abstract— **In this paper, we consider the semi-global regulation of output synchronization problem for heterogeneous networks of invertible linear agents subject to actuator saturation. That is, we regulate the output of each agent according to an *a priori* specified reference model. The network communication infrastructure provides each agent with a linear combination of its own output relative to that of neighboring agents, and it allows the agents to exchange information about their own internal observer estimates, while some agents have access to their own outputs relative to the reference trajectory.**

I. Introduction

The synchronization problem in a network has received substantial attention in recent years (see [1], [12], [18], [30] and references therein). Active research is being conducted in this context and numerous results have been reported in the literature; to name a few, see [7], [11], [12], [13], [15], [16], [17], [23], [24], [25].

Much of the attention has been devoted to achieving *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), where each agent has access to a linear combination of its own state relative to that of neighboring agents (e.g., [10], [11], [12], [16], [17], [19], [20], [21], [24], [32]). A more realistic case—that is, each agent receives a linear combination of its own output relative to that of neighboring agents—has been considered in [7], [13], [14], [25], [26]. A key idea in the work of [7], which was expanded upon by Yang, Stoorvogel, and Saberi [34], is the development of a distributed observer. This observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many results on the synchronization problem are rooted in the seminal work [28], [29].

A. Heterogeneous networks and output synchronization

Recent activities in the synchronization literature have been focused on achieving synchronization for heterogeneous networks (i.e., networks where the agent models are non-identical). This problem is challenging and only some results are available; see, for instance, [2], [5], [6], [9], [27], [31].

In heterogeneous networks, the agents' states may have different dimensions. In this case, the state synchronization is not even properly defined, and it is more natural to aim for *output synchronization*—that is, asymptotic agreement on some output from each agent. Chopra and Spong [2] studied output synchronization for weakly minimum-phase nonlinear systems of relative degree one, using a pre-feedback to create a single-integrator system with decoupled zero dynamics. Kim, Shim, and Seo [6] considered the output synchronization problem for uncertain single-input single-output, minimum-phase linear systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. The authors have considered in [33] the output synchronization problem for right-invertible linear agents, using pre-compensators and an observer-based pre-feedback within each agent to yield a network of agents which are to a large extent identical.

B. Introspective versus non-introspective agents

The designs mentioned in Section I-A generally rely on some sort of self-knowledge that is separate from the information transmitted over the network. More specifically, the agents may be required to know their own states or their own outputs. In [3], [4], we refer to agents that possess this type of self-knowledge as *introspective* agents to distinguish them from *non-introspective* agents—that is, agents that have no knowledge about their own states or outputs separate from what is received via the network.

To our best knowledge, the only result besides [3], [4] that clearly applies to heterogeneous networks of non-introspective agents is by Zhao, Hill and Liu [35]. However, the agents are assumed to be passive—a strict requirement that, among other things, requires that the agents are weakly minimum-phase and of relative degree one.

C. Contributions of this paper

The *regulation of output synchronization* problem, where the objective is not only to achieve output synchronization, but to make the synchronization trajectory follow an *a priori* given reference trajectory generated by an arbitrary autonomous exosystem, has been considered in [4]. In [4], we assume that the agents in the network are non-introspective except for some of the agents, which know their own outputs relative to the reference trajectory. However, we do not have any constraints on the magnitude of the agent's input. In the real world, every physically conceivable actuator has bounds on its input, and thus actuator saturation is a common phenomenon. In this paper, we extend the results in [4] to the

The work of Tao Yang and Ali Saberi is partially supported by National Science Foundation grant ECS-0901137, and NAVY grants ONR KKK777SB001 and ONR KKK760SB0012.

[1]ACCESS Linnaeus Centre, Royal Institute of Technology (KTH), Sweden. E-mail: taoyang.work@gmail.com.

[2]Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500AE Enschede, The Netherlands. E-mail: A.A.Stoorvogel@utwente.nl.

[3]School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A. E-mail: saberi@eecs.wsu.edu, grip@ieee.org.

case where all the agents are subject to actuator saturation, which introduces significant complexities in terms of the analysis and design.

II. Problem Formulation and Main Result

A. Problem Formulation

Consider a network of N multiple-input multiple-output invertible agents of the form

$$\dot{x}_i = A_i x_i + B_i \sigma(u_i), \tag{1a}$$

$$y_i = C_i x_i + D_i \sigma(u_i), \tag{1b}$$

for $i \in \{1, \ldots, N\}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^p$, and

$$\sigma(u_i) = [\sigma_1(u_{i,1}), \ldots, \sigma_1(u_{i,p})]',$$

where $\sigma_1(u)$ is the standard saturation function

$$\sigma_1(u) = \operatorname{sgn}(u) \min\{1, |u|\},$$

and where the quadruple (A_i, B_i, C_i, D_i) is invertible.

The network provides each agent with a linear combination of its own output relative to that of other agents. In particular, each agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{2}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ with $i, j \in \{1, \ldots, N\}$. This network can be described by a weighted directed graph (digraph) $\mathscr{G}$ with nodes corresponding to the agents in the network and edges with weight given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ means that there exists an edge with weight a_{ij} from agent j to agent i.

We also define a matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. The matrix G, known as the *weighted Laplacian* matrix of the digraph $\mathscr{G}$, has the property that the sum of the coefficients on each row is equal to zero. In terms of the coefficients g_{ij} of G, ζ_i given by (2) can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j. \tag{3}$$

In addition to ζ_i given by (3), we assume that the agents exchange information about their own internal estimates via the same network. That is, agent i has access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{N} g_{ij} \eta_j, \tag{4}$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced internally by agent j. This value will be specified as we proceed with the design.

Our goal is to regulate the outputs of all agents towards an *a priori* specified reference trajectory $y_r(t)$, generated by an arbitrary autonomous exosystem

$$\dot{\omega} = S\omega, \quad \omega(0) = \omega_0 \in \Omega_0, \tag{5a}$$

$$y_r = C_r \omega, \tag{5b}$$

where $\omega \in \mathbb{R}^r$, $y_r \in \mathbb{R}^p$, and Ω_0 is a compact set of possible initial conditions for the exosystem. That is, for each agent $i \in \{1, \ldots, N\}$, we wish to achieve $\lim_{t\to\infty}(y_i - y_r) = 0$. Equivalently, we wish to regulate the synchronization error variable $e_i := y_i - y_r$ to zero asymptotically, where the dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \omega \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{6a}$$

$$e_i = \begin{bmatrix} C_i & -C_r \end{bmatrix} \begin{bmatrix} x_i \\ \omega \end{bmatrix} + D_i \sigma(u_i). \tag{6b}$$

In order to achieve our goal, in addition to ζ_i given by (3) and $\hat{\zeta}_i$ given by (4) provided by the network, it is clear that a non-empty subset of agents should observe its output relative to the reference trajectory y_r generated by (5) in order for the network of agents to follow the reference trajectory. Specifically, let $\mathscr{I} \subset \{1, \ldots, N\}$ denote such a subset. Then, each agent $i \in \{1, \ldots, N\}$ has access to the quantity $\psi_i = \iota_i(y_i - y_r)$, where $\iota_i = 1$ if $i \in \mathscr{I}$, and $\iota_i = 0$ if $i \notin \mathscr{I}$.

Clearly, we need to restrict the initial conditions of the exosystem since, due to the input saturation, the agents will only be able to track a limited set of reference trajectories. This is formulated in the above by assuming that $\omega(0) \in \Omega_0$ with the set Ω_0 known a priori. Regarding the initial conditions of the agents, we would ideally like to design a controller that achieves $\lim_{t\to\infty} e_i(t) = 0$ for all initial conditions subject to $\omega(0) \in \Omega_0$, a problem that can be referred to as *global regulation of output synchronization*. However, from the literature on linear systems subject to actuator saturation, we know that global regulation of output synchronization in general requires nonlinear controllers. In this paper, we would like to use linear controllers of the form:

$$\dot{x}_i^c = A_{i,c} x_i^c + B_{i,c} \begin{bmatrix} \zeta_i \\ \hat{\zeta}_i \\ \psi_i \end{bmatrix}, \tag{7a}$$

$$u_i = C_{i,c} x_i^c, \quad \forall i \in \{1, \ldots, N\}, \tag{7b}$$

where $x_i^c \in \mathbb{R}^{c_i}$ is the state of the controller for agent i. Thus, we restrict attention to the *semi-global regulation of output synchronization* problem, which is defined as follows.

Problem 1: Consider a network of N agents as given by (1) and the reference model given by (5) with initial conditions in an *a priori* given compact set $\Omega_0 \subset \mathbb{R}^r$. The *semi-global regulation of output synchronization problem* is to find, if possible, for certain integers c_i, $i \in \{1, \ldots, N\}$ a family of controllers of the form (7) parameterized in a parameter ε such that for any arbitrarily large bounded sets $\mathscr{X}_i \subset \mathbb{R}^{n_i}$ and $\mathscr{P}_i \subset \mathbb{R}^{c_i}$, $i \in \{1, \ldots, N\}$, there exists ε small enough for which

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall i \in \{1, \ldots, N\}, \tag{8}$$

for all initial conditions $x_i(0) \in \mathscr{X}_i$, $x_i^c(0) \in \mathscr{P}_i$, and $\omega(0) \in \Omega_0$.

B. Assumptions

In this section, we present the assumptions about the network topology, the individual agents, and the reference model for solving the semi-global regulation of output synchronization problem as defined in Problem 1.

Assumption 1: Every node of the digraph $\mathscr{G}$ is a member of a directed tree whose root is contained in $\mathscr{I}$.

Remark 1: It is possible for $\mathscr{I}$ to consist of a single node, in which case Assumption 1 requires this node to be the root of a directed spanning tree of $\mathscr{G}$.

Assumption 2: For each agent $i \in \{1,\dots,N\}$ as given in (1)

1) all the eigenvalues of A_i are in the closed left-half complex plane;
2) the pair (A_i, B_i) is stabilizable; and
3) the pair (C_i, A_i) is observable;

Remark 2: Conditions 2 and 3 are natural assumptions. Condition 1 is a necessary condition, since if A_i has one observable eigenvalue in the open right-half complex plane for some $i \in \{1,\dots,N\}$, then for sufficiently large initial conditions $x_i(0)$, the output of that system y_i will be exponentially growing, and the bounded input $\sigma(u_i)$ can influence this exponentially growing signal only in a limited sense. Therefore, we cannot guarantee that this output will track y_r.

Assumption 3: For the reference model (5),

1) the pair (C_r, S) is observable;
2) all the eigenvalues of S are in the closed right-half complex plane; and
3) the matrix S is neutrally stable.

Remark 3: Condition 1 is a natural assumption. Condition 2 is made without loss of generality because asymptotically stable modes vanish asymptotically, and therefore they play no role asymptotically. Condition 3 is reasonable since the output of an agent cannot be expected to track exponentially growing signals with a bounded input. Polynomially growing reference signals can be easily included but it requires very restrictive solvability conditions in case of input saturation and hence, for ease of presentation, we have excluded this case.

Assumption 4: The equations

$$\Pi_i S = A_i \Pi_i + B_i \Gamma_i, \tag{9a}$$
$$C_r = C_i \Pi_i + D_i \Gamma_i, \tag{9b}$$

commonly known as the *regulator equations* are solvable with respect to $\Pi_i \in \mathbb{R}^{n_i \times r}$ and $\Gamma_i \in \mathbb{R}^{p \times r}$, and there exists a $\delta > 0$ such that for each agent $i \in \{1,\dots,N\}$,

$$\|\Gamma_i \omega(t)\|_\infty \le 1 - \delta, \tag{10}$$

for all $t > 0$ and all $\omega(t)$ with $\omega(0) \in \Omega_0$.

Remark 4: Note that if the regulator equations (9) have a solution, then the solution is unique, as a consequence of the invertibility of the quadruple (A_i, B_i, C_i, D_i). Therefore, one can easily verify (10).

C. Necessity of Assumption 4

Assumptions 1, 2, and 3 are natural as discussed in Remarks 2 and 3. On the other hand, Assumption 4 is critical. Essentially, this assumption is necessary for solving the semi-global regulation of output synchronization problem as defined in Problem 1. The following lemma shows this fact and gives the necessary condition for solving Problem 1.

Lemma 1: Assume that Ω_0 contains the origin in its interior. Then for any initial condition $\omega(0) \in \Omega_0$, there exist initial conditions $x_i(0)$ and an input $u_i(t)$ that leads to $e_i(t) \to 0$ as $t \to \infty$ only if the regulator equations (9) are solvable, and moreover the solution of the regulator equation must satisfy

$$\|\Gamma_i \omega(t)\|_\infty \le 1 \tag{11}$$

for all $t > 0$.

Proof: We have omitted the proof due to the space limitation. ■

D. Main Result

Theorem 1: Consider a network of N agents as given by (1) and the reference model given by (5). Let Assumptions 1, 2, 3, and 4 hold. Then the semi-global regulation of output synchronization problem as defined in Problem 1 is solvable.

Proof: The proof of Theorem 1 is given in Section III by explicit construction of a controller for each agent. ■

III. Design of control law for each agent

In this section, we describe the construction of a controller for each agent to solve the semi-global regulation of output synchronization problem as defined in Problem 1. The construction is carried out in three steps.

In Step 1, we construct a new state $\bar{x}_i$, via a transformation of x_i and ω, such that the dynamics of the synchronization error variable e_i can be described by equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i \sigma(u_i) := \begin{bmatrix} A_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{12a}$$
$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i \sigma(u_i) := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i \sigma(u_i). \tag{12b}$$

The purpose of this state transformation is to reduce the dimension of the model underlying e_i—the dimension of $\bar{x}_i$ is generally lower than that of $[x_i', \omega']'$—by removing redundant modes that have no effect on e_i. In particular, the model (6) may be unobservable, but the model (12) is always observable.

In Step 2, we construct a low-gain state feedback for u_i assuming $\bar{x}_i$ is known. This feedback is parameterized in ε and regulates e_i to zero for any arbitrarily large bounded set of initial conditions of the agent's models by choosing the low-gain parameter ε sufficiently small. Moreover, by making the low-gain parameter ε small enough, we can guarantee that the amplitude of the control law is less than any given $\alpha > 0$. Since the agent i has neither the internal state x_i nor the state ω of the exosystem available, this controller is not directly implementable. This brings us to Step 3 of the design.

In Step 3, we follow the procedure as given in our previous paper [4], that is, we construct a decentralized high-gain observer that makes an estimate of $\bar{x}_i$ available to agent i. However, as we shall see later, our state feedback design and high-gain observer are coupled. This will be illustrated in Section III-A.

A. Design procedure for agent i

Step 1: State transformation

Let O_i be the observability matrix corresponding to the system (6).

$$\mathscr{O}_i = \begin{bmatrix} C_i & -C_r \\ \vdots & \vdots \\ C_i A_i^{n_i+r-1} & -C_r S^{n_i+r-1} \end{bmatrix}.$$

Let q_i denote the dimension of the null space of matrix $\mathscr{O}_i$, and define $r_i = r - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{r \times q_i}$ such that

$$\mathscr{O}_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \quad \operatorname{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i.$$

Since the pair (C_i, A_i) and the pair (C_r, S) are observable, it is easy to see that Λ_{iu} and Φ_{iu} have full column rank (see [4, Appendix A]). Let therefore Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{r \times r}$ are nonsingular.

From the proof of [3, Lemma 2], we know that

$$S\Phi_i = \Phi_i R_i, \tag{13}$$

where

$$R_i = \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix}.$$

Since S is anti-Hurwitz stable and neutrally stable, we know that S is diagonalizable, and hence R_i is diagonalizable. This implies that R_i has r independent right eigenvectors. Let $v_{i,1}, \cdots, v_{i,r}$ be r independent right eigenvectors of R_i, such that

$$v_{i,j} = \begin{bmatrix} \tilde{v}_{i,j} \\ 0 \end{bmatrix}$$

for $j = 1, \ldots, q_i$, where $\tilde{v}_{i,j}$ are right eigenvectors of U_i. In that case we choose $V_{i11} \in \mathbb{R}^{q_i \times q_i}$ such that

$$\operatorname{Im} V_{i11} = \operatorname{span}\{\operatorname{re} \tilde{v}_{i,j}, \operatorname{im} \tilde{v}_{i,j} \mid j = 1, \cdots, q_i\}^1$$

and we choose $V_{i12} \in \mathbb{R}^{q_i \times r_i}$ and $V_{i22} \in \mathbb{R}^{r_i \times r_i}$ such that

$$\operatorname{Im} \begin{bmatrix} V_{i12} \\ V_{i22} \end{bmatrix} = \operatorname{span}\{\operatorname{re} v_{i,j}, \operatorname{im} v_{i,j} \mid j = q_i + 1, \cdots, r\}.$$

We then construct:

$$V_i = \begin{bmatrix} V_{i11} & V_{i12} \\ 0 & V_{i22} \end{bmatrix}.$$

It can be easily verified that $\operatorname{span}\{\operatorname{re} v_{i,j}, \operatorname{im} v_{i,j}\}$ is an invariant subspace of R_i for any $j = 1, \ldots r$. This implies:

$$R_i V_i = V_i \begin{bmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{bmatrix}. \tag{14}$$

One way of choosing the matrix V_i is choosing

$$\begin{bmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{bmatrix}$$

[1]For a given vector $v \in \mathbb{C}^n$, $\operatorname{re} v \in \mathbb{R}^n$ and $\operatorname{im} v \in \mathbb{R}^n$ denote respectively vectors whose entries are the real part and imaginary part of the vector v.

to be the real Jordan form of R_i ordered in such a way that Λ_{i1} is the real Jordan form of U_i.

From (14), we obtain that

$$V_{i11}^{-1} U_i V_{i11} = \Lambda_{i1}, \quad V_{i22}^{-1} R_{i22} V_{i22} = \Lambda_{i2}, \tag{15}$$

and

$$U_i V_{i12} - V_{i12} \Lambda_{i2} = -R_{i12} V_{i22}. \tag{16}$$

We then define

$$\bar{\Phi}_i := [\bar{\Phi}_{iu}, \bar{\Phi}_{io}] = \Phi_i \begin{bmatrix} I_{q_i} & V_{i12} V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix}. \tag{17}$$

We then define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i + r_i}$ as

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{i1} \\ \bar{x}_{i2} \end{bmatrix} := \begin{bmatrix} x_i - \Lambda_i M_i \bar{\Phi}_i^{-1} \omega \\ N_i \bar{\Phi}_i^{-1} \omega \end{bmatrix},$$

where $M_i \in \mathbb{R}^{n_i \times r}$ and $N_i \in \mathbb{R}^{r_i \times r}$ are defined as

$$M_i = \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 & I_{r_i} \end{bmatrix}.$$

Note that the system (6) can be transformed into the system (12), with a block upper-triangular structure if we use the transformation Φ_i as shown in [4]. However, with the matrix $\bar{\Phi}_i$ given by (17), which is a special case of the transformation previously used in [4], everything from our previous results still holds. Moreover, the system (12) has a block-diagonal structure. The following lemmashows this.

Lemma 2: The synchronization error variable e_i is governed by dynamical equations of (12), where the pair $(\bar{C}_i, \bar{A}_i)$ is observable and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of S.

Proof: We have omitted the proof due to the space limitation. ■

Remark 5: If the unforced system for an agent i is the same as the exosystem, i.e., if $C_i = C_r$ and $A_i = S$, then it is easy to see that the dynamics of system (12) reduces to the dynamics of system (1).

Step 2: State feedback control design

For any arbitrarily large bounded set $\mathscr{X}_i$, we design a controller as a function of $\bar{x}_i$ such that $\lim_{t\to\infty} e_i(t) = 0$ for all $x_i(0) \in \mathscr{X}_i$ and $\omega(0) \in \Omega_0$. Consider the following regulator equations with unknowns $\Pi_i^r \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i^r \in \mathbb{R}^{p \times r_i}$ for system (12)

$$\Pi_i^r \bar{A}_{i22} = A_i \Pi_i^r + B_i \Gamma_i^r, \tag{18a}$$

$$\bar{C}_{i2} = C_i \Pi_i^r + D_i \Gamma_i^r. \tag{18b}$$

The following lemma shows that the regulator equations (18) are solvable if and only if the regulator equations (9) are solvable, and gives the mapping between the solutions of the two regulator equations. Note that if the regulator equations (18) (or the regulator equations (9)) have a solution, then it is unique due to the invertibility of the quadruple (A_i, B_i, C_i, D_i).

Lemma 3: If (Π_i^r, Γ_i^r) is the solution of the regulator equations (18), then (Π_i, Γ_i) given as

$$\Pi_i = \Pi_i^r N_i \bar{\Phi}_i^{-1} + \Lambda_i M_i \bar{\Phi}_i^{-1}, \quad \Gamma_i = \Gamma_i^r N_i \bar{\Phi}_i^{-1} \tag{19}$$

is the solution of the regulator equations (9). On the other hand, if (Π_i, Γ_i) is the solution of the regulator equations (9), then (Π_i^r, Γ_i^r) given as

$$\Pi_i^r = \Pi_i \bar{\Phi}_{io}, \quad \Gamma_i^r = \Gamma_i \bar{\Phi}_{io} \tag{20}$$

is the solution of the regulator equations (18).

Proof: We have omitted the proof due to the space limitation. ■

Remark 6: In view of Lemma 3 and (10) of Assumption 4, we see that $\|\Gamma_i^r \bar{x}_{i2}\|_\infty = \|\Gamma_i \omega\|_\infty \leq 1 - \delta$.

Since agent i is subject to actuator saturation, we design the state feedback controller by using a *low-gain* technique, which is widely used for the semi-global stabilization problem for linear systems subject to actuator saturation, see for instance, [8], [22]. There exist in the literature several low-gain design algorithms. For conceptual clarity, we use here the one based on the solution of a continuous-time algebraic Riccati equation, parameterized in a low-gain parameter $\varepsilon \in (0,1]$. More specifically, we form a family of parameterized state feedback gain matrices $F_{i,\varepsilon}$ for $\bar{x}_{i1}$ as

$$F_{i,\varepsilon} = -B_i' P_{i,\varepsilon},$$

where $P_{i,\varepsilon} = P_{i,\varepsilon}' > 0$ is the unique solution of the continuous-time algebraic Riccati equation defined as

$$P_{i,\varepsilon} A_i + A_i' P_{i,\varepsilon} - P_{i,\varepsilon} B_i B_i' P_{i,\varepsilon} + \varepsilon I_{n_i} = 0. \tag{21}$$

It follows from Lemma 3 and and Condition 1 of Assumption 4 that the regulator equations (18) have a unique solution (Π_i^r, Γ_i^r). We use the unique (Π_i^r, Γ_i^r) and the feedback gain matrix $F_{i,\varepsilon}$ to define a family of parameterized state feedback controllers in terms of $\bar{x}_i$ as

$$u_i = \begin{bmatrix} F_{i,\varepsilon} & \Gamma_i^r - F_{i,\varepsilon}\Pi_i^r \end{bmatrix} \bar{x}_i. \tag{22}$$

Then for any given arbitrarily large bounded set of initial conditions, there exists an $\varepsilon^* \in (0,1]$, such that for all $\varepsilon \in (0, \varepsilon^*]$, the family of linear state feedback controllers of the form (22) ensures that $\lim_{t\to\infty} e_i(t) = 0$ for all initial conditions belong to the given arbitrarily large bounded set and $\omega(0) \in \Omega_0$. This is a well known result, see [22, Theorem 3.3.2].

Remark 7: If the unforced system for an agent i is the same as the exosystem, i.e., if $C_i = C_r$ and $A_i = S$, then it is easy to see that $\Pi_i = I$ and $\Gamma_i = 0$ is the solution of regulator equations (9). Thus, Assumption 4 is always satisfied for that agent.

Step 3: Observer-based implementation

Following the design procedure given in the proof of [22, Theorem 3.3.4], one can obtain, for a given set of initial conditions, suitable state feedback controllers for which input saturation is not active. This is done by properly choosing the low-gain parameter ε. Then such a state feedback law must be implemented by a suitable designed distributed observer. This will be done next.

We will design a high-gain decentralized observer to produce an estimate of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$. We follow the procedure as given in our previous paper [4], to be self-contained, we reproduce the design here.

Let $\bar{n}$ denotes the maximum order among the all the systems (12) for $i \in \{1, \dots, N\}$, that is, $\bar{n} = \max_{i=1,\dots,n}(n_i + r_i)$. Define $\chi_i = T_i \bar{x}_i$, where

$$T_i = \begin{bmatrix} \bar{C}_i \\ \vdots \\ \bar{C}_i \bar{A}_i^{\bar{n}-1} \end{bmatrix}.$$

Note that T_i is injective since the pair $(\bar{C}_i, \bar{A}_i)$ is observable, which implies that $T_i' T_i$ is nonsingular.

In term of χ_i, we can write the system equations

$$\dot{\chi}_i = (\mathscr{A} + \mathscr{L}_i)\chi_i + \mathscr{B}_i \sigma(u_i), \quad \chi_i(0) = T_i \bar{x}_i(0), \tag{23a}$$
$$e_i = \mathscr{C}\chi_i + \mathscr{D}_i \sigma(u_i), \tag{23b}$$

where

$$\mathscr{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} I_p & 0 \end{bmatrix},$$
$$\mathscr{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathscr{B}_i = T_i \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \mathscr{D}_i = D_i,$$

and where $L_i = \bar{C}_i \bar{A}_i^{\bar{n}} (T_i' T_i)^{-1} T_i'$.

We define the matrix $\bar{G} = G + \operatorname{diag}(\iota_1, \dots, \iota_N)$. It then follows from [4, Lemma 7] that all the eigenvalues of $\bar{G}$ are in the open right-half complex plane. Next, define $\tau = \min_{i=1,\dots,N} \operatorname{Re}(\lambda_i(\bar{G})) > 0$.

The pair $(\mathscr{C}, \mathscr{A})$ is always observable; hence, we can define a matrix $\mathscr{P} = \mathscr{P}' > 0$ as the unique solution of the algebraic Riccati equation

$$\mathscr{A}\mathscr{P} + \mathscr{P}\mathscr{A}' - \tau \mathscr{P}\mathscr{C}'\mathscr{C}\mathscr{P} + I_{\bar{n}p} = 0. \tag{24}$$

We construct the observer

$$\begin{aligned} \dot{\hat{\chi}}_i = {} & (\mathscr{A} + \mathscr{L}_i)\hat{\chi}_i + \mathscr{B}_i \sigma(u_i) + S(\ell)\mathscr{P}\mathscr{C}'(\zeta_i - \hat{\zeta}_i) \\ & + S(\ell)\mathscr{P}\mathscr{C}'(\psi_i - \iota_i(\mathscr{C}\hat{\chi}_i + D_i\sigma(u_i))), \end{aligned} \tag{25a}$$
$$\hat{\bar{x}}_i = (T_i' T_i)^{-1} T_i' \hat{\chi}_i, \tag{25b}$$

where $S(\ell) = \text{blk diag}(I_p \ell, I_p \ell^2, \dots, I_p \ell^{\bar{n}})$ and $\ell > 1$ is a high-gain parameter.

Based on the observer estimate, we define the variable $\eta_i = \mathscr{C}\hat{\chi}_i + \mathscr{D}_i \sigma(u_i)$ to be shared with the other agents via the network communication infrastructure as described in Section II-A, and the observer-based control law

$$u_i = \begin{bmatrix} F_{i,\varepsilon} & \Gamma_i^r - F_{i,\varepsilon}\Pi_i^r \end{bmatrix} \hat{\bar{x}}_i. \tag{26}$$

Together, the observers for agents $i \in \{1, \dots, N\}$ form a distributed observer parameterized by a high-gain parameter ℓ. It has been shown in [4, Lemma 4] that the estimation errors dynamics are globally exponentially stable, that is, $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$, by choosing the high-gain parameter ℓ sufficiently large.

Remark 8: If all the agents have the same dynamics, it is not necessary to design an observer based on the high-order system (23) and one can design an observer based on the original system (12).

In summary, for any given arbitrarily large bounded sets $\mathscr{X}_i \subset \mathbb{R}^{n_i}$ and $\mathscr{P}_i \subset \mathbb{R}^{p\bar{n}}$, there exist ε^* with the property that for any $\varepsilon \in (0, \varepsilon^*]$ there exists ℓ^* such that for $\ell \geq \ell^*$, the observer-based implementation (25) and (26), ensure that

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall i \in \{1,\ldots,N\}, \tag{27}$$

for all initial conditions $x_i(0) \in \mathscr{X}_i$, $\hat{\chi}_i(0) \in \mathscr{P}_i$, and $\omega(0) \in \Omega_0$.

B. Comparison with the case where the agents have no actuator magnitude constraints

Let us make a few comments to compare our result to the case where the agents do not have actuator saturation.

- The regulator equations (9) have to be solvable for the case with actuator magnitude constraints. In our previous work for the case without saturation we assumed existence of a solution of the regulator equations but in that case this existence is not necessary.
- For the case with actuator magnitude constraints, we only achieve semi-global regulation of output synchronization.
- For the case with actuator magnitude constraints, it is required that all the eigenvalues of agents' system matrices are in the closed left-half complex plane.
- For the case with actuator magnitude constraints, we have constraints on the size of the synchronized output trajectory as given by (10).

REFERENCES

[1] H. Bai, M. Arack, and J. Wen, *Cooperative Control Design: A Systematic, Passivity-Based Approach*, Communications and Control Engineering Series, Springer Verlag, New York, 2011.

[2] N. Chopra and W. Spong, "Output synchronization of nonlinear systems with relative degree one", in Recent advances in learning and control, V.D. Blondel, S.P. Boyd, and H. Kimura, eds., vol. 371 of Lecture notes in control and information sciences, Springer Verlag, London, 2008, pp. 51–64.

[3] H.F.Grip, T. Yang, A. Saberi, and A.A. Stoorvogel, "Decentralized control for output synchronization in heterogeneous networks of non-introspective agents", in Proc. American Control Conference, Montréal, Canada, 2012, pp. 812–819.

[4] H.F.Grip, T. Yang, A. Saberi, and A.A. Stoorvogel, "Output synchronization for heterogeneous networks of non-introspective agents", Automatica, 48(10), 2012, pp. 2444–2453.

[5] D.J. Hill and J. Zhao, "Global synchronization of complex dynamical networks with non-identical nodes", in Proc. 47th CDC, Cancun, Mexico, 2008, pp. 817–822.

[6] H. Kim, H. Shim, and J.H. Seo, "Output consensus of heterogeneous uncertain linear multi-agent systems", IEEE Trans. Aut. Contr., 56(1), 2011, pp. 200–206.

[7] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint", IEEE Trans. Circ. & Syst.-I Regular papers, 57(1), 2010, pp. 213–224.

[8] Z. Lin and A. Saberi, "Semi-global exponential stabilization of linear systems subject to "input saturation" via linear feedbacks", Syst. & Contr. Letters, 21(3), 1993, pp. 225–239.

[9] J. Mei, W. Ren, and G. Ma, "Distributed Coordinated Tracking With a Dynamic Leader for Multiple Euler-Lagrange Systems", IEEE Trans. Aut. Contr., 56(6), 2011, pp. 1415–1421.

[10] R. Olfati-Saber, J.A. Fax, and R.M. Murray, "Consensus and cooperation in networked multi-agent systems", Proc. of the IEEE, 95(1), 2007, pp. 215–233.

[11] R. Olfati-Saber and R.M. Murray, "Agreement problems in networks with direct graphs and switching topology", in Proc. 42nd CDC, Maui, Hawaii, 2003, pp. 4126–4132.

[12] R. Olfati-Saber and R.M. Murray, "Consensus problems in networks of agents with switching topology and time-delays", IEEE Trans. Aut. Contr., 49(9), 2004, pp. 1520–1533.

[13] A.Y. Pogromsky and H. Nijmeijer, "Cooperative oscillatory behavior of mutually coupled dynamical systems", IEEE Trans. Circ. & Syst.-I Fundamental theory and applications, 48(2), 2001, pp. 152–162.

[14] A.Y. Pogromsky, G. Santoboni, and H. Nijmeijer, "Partial synchronization: from symmetry towards stability", Physica D, 172(1-4), 2002, pp. 65–87.

[15] W. Ren, "On consensus algorithms for double-integrator dynamics", IEEE Trans. Aut. Contr., 53(6), 2008, pp. 1503–1509.

[16] W. Ren and E. Atkins, "Distributed multi-vehicle coordinate control via local information", Int. J. Robust & Nonlinear Control, 17(10-11), 2007, pp. 1002–1033.

[17] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control", IEEE Control Systems Magazine, 27(2), 2007, pp. 71–82.

[18] W. Ren and Y. Cao, *Distributed Coordination of Multi-agent Networks*, Communications and Control Engineering Series, Springer Verlag, London, 2011.

[19] W. Ren, K. L. Moore, and Y. Chen, "High-order and model reference consensus algorithms in cooperative control of multi-vehicle systems", Journal of Dynamic Systems, Measurement, and Control, 129(5), 2007, pp. 678–688.

[20] S. Roy, A. Saberi, and K. Herlugson, "Formation and alignment of distributed sensing agents with double-integrator dynamics", in Sensor Network Operations, S. Phoha, T.F. La Porta, and C. Griffin, eds., Wiley-IEEE Press, 2006.

[21] S. Roy, A. Saberi, and K. Herlugson, "A control-theoretic perspective on the design of distributed agreement protocols", Int. J. Robust & Nonlinear Control, 17(10-11), 2007, pp. 1034–1066.

[22] A. Saberi, A.A. Stoorvogel, and P. Sannuti, *Control of linear systems with regulation and input constraints*, Communication and Control Engineering Series, Springer Verlag, Berlin, 2000.

[23] J.H. Seo, H. Shim, and J. Back, "Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach", Automatica, 45(11), 2009, pp. 2659–2664.

[24] S.E. Tuna, "LQR-based coupling gain for synchronization of linear systems", Available: arXiv:0801.3390v1, 2008.

[25] S.E. Tuna, "Synchronizing linear systems via partial-state coupling", Automatica, 44(8), 2008, pp. 2179–2184.

[26] S.E. Tuna, "Conditions for synchronizability in arrays of coupled linear systems", IEEE Trans. Aut. Contr., 55(10), 2009, pp. 2416–2420.

[27] P. Wieland, R. Sepulchre, and F. Allgöwer, "An internal model principle is necessary and sufficient for linear output synchronization", Automatica, 47(5), 2011, pp. 1068–1074.

[28] C.W. Wu and L.O. Chua, "Application of graph theory to the synchronization in an array of coupled nonlinear oscillators", IEEE Trans. Circ. & Syst.-I Fundamental theory and applications, 42(8), 1995, pp. 494–497.

[29] C.W. Wu and L.O. Chua, "Application of Kronecker products to the analysis of systems with uniform linear coupling", IEEE Trans. Circ. & Syst.-I Fundamental theory and applications, 42(10), 1995, pp. 775–778.

[30] C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*, World Scientic: Singapore, Singapore, 2007.

[31] J. Xiang and G. Chen, "On the V-stability of complex dynamical networks", Automatica, 43(6), 2007, pp. 1049–1057.

[32] T. Yang, S. Roy, Y. Wan, and A. Saberi, "Constructing consensus controllers for networks with identical general linear agents", Int. J. Robust & Nonlinear Control, 21(11), 2011, pp. 1237–1256.

[33] T. Yang, A. Saberi, A.A. Stoorvogel, and H.F. Grip, "Output consensus for networks of non-identical introspective agents", in Proc. 50th CDC, Orlando, FL, 2011, pp. 1286–1292.

[34] T. Yang, A.A. Stoorvogel, and A. Saberi, "Consensus for multi-agent systems — synchronization and regulation for complex networks", in American Control Conference, San Francisco, CA, 2011, pp. 5312–5317.

[35] J. Zhao, D.J. Hill, and T. Liu, "Passivity-based output synchronization of dynamical network with non-identical nodes", in Proc. 49th CDC, Atlanta, GA, 2010, pp. 7351–7356.