Examples of Applications of Potential Functions in Problem Solving (Web Appendix to the Paper)

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1 Introduction

Potential functions may be exploited to formulate various phenomena. This is particularly of interest because it offers a unified notion for problem solving and for developing control strategies. Moreover, many problems involving vector variables can be conveniently solved using the notion of potential. In this approach, vector variables are replaced with scalar variables, which are easier and less time-consuming to deal with. In this section, example applications of potential functions in engineering are described.

2 Potential Functions for Problems in Electric Fields

Fig. 1 shows four point charges $q_1, q_2, q_3,$ and q_4. It is desired to calculate the electric field at point P. This can be done based on either of the following two approaches:

(i) Direct calculation of the electric field: In this method, the electric fields due to each of the four point charges at P are calculated. Applying superposition, the net field at P is the *vector* sum of these four electric field terms.

The electric field due to charge q_i at P is

$$E_{Pi} = \frac{1}{4\pi\epsilon_0}\frac{q_i\hat{r}_i}{r_i^2} = \frac{1}{4\pi\epsilon_0}\frac{q_i r_i}{|r_i|^3}, \tag{1}$$

where r_i is the vector from q_i to P, $\hat{r_i}$ is the unit vector of r_i, q_i is the amount of charge, and ϵ_0 is the vacuum permittivity. The net field at P is

$$E_P = \sum_{i=1}^{4} E_{Pi} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{4}\frac{q_i r_i}{|r_i|^3}. \tag{2}$$

(ii) Calculation of the electric field from the electric potential: In this method, the electric potential of P due to each of the four point charges is calculated. The net potential at P is the *scalar* sum of the individual terms. The electric field is then determined by calculating the gradient of the potential.

The potential at P due to q_i is

$$V_i = \frac{1}{4\pi\epsilon_0}\frac{q_i}{|r_i|}, \tag{3}$$

and the net potential at P due to all charges is the scalar sum of V_i terms:

$$V = \sum_{i=1}^{4} V_i. \tag{4}$$

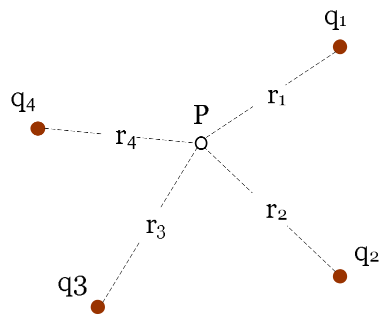

Fig. 1. Calculation of the electric field at P due to four point charges.

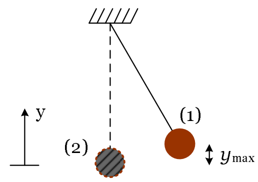

Fig. 2. A pendulum at two instants.

The field at P is calculated from the gradient of the electric potential as

$$\begin{aligned} E &= -\nabla V \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{4}\frac{q_i}{(x_i^2+y_i^2)\sqrt{x_i^2+y_i^2}}\begin{bmatrix}x_i\\ y_i\end{bmatrix} \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{4}\frac{q_i}{r_i^2}\hat{r_i}. \end{aligned} \tag{5}$$

Since performing scalar summation is faster and simpler than vector summation, it is more convenient to calculate the field at P by the method of potential.

3 Potential Functions for Problems in Dynamics of Motion

Fig. 2 illustrates an example: a hanging pendulum in the gravitational field of Earth at two distinct instants. It is desired to find a relationship between the maximum speed of the pendulum and its maximum vertical deviation.

At position 1, the pendulum is at its maximum vertical displacement. At the moment immediately before the pendulum reaches position 1, the pendulum is going upward, while at the moment immediately after, the pendulum is going downward. Therefore, the sign of the velocity of the pendulum changes at 1, meaning the velocity momentarily becomes zero. At position 2, the pendulum is at its lowest height. If 2 is selected as the base height for calculation of the gravitational potential energy, the potential of the pendulum at 2 is zero. Assuming an isolated system with no friction and stray masses, the energy of the system is constant at all positions, including 1 and 2.

The total energy of the pendulum consists of a term for gravitational potential energy and a term for mechanical kinetic energy. The total energy of the pendulum at 1, assuming a uniform gravitational potential field, is

$$U_1 = U_{1P} + U_{1K} = \Big(mg_0 y_{\max}\Big) + 0, \tag{6}$$

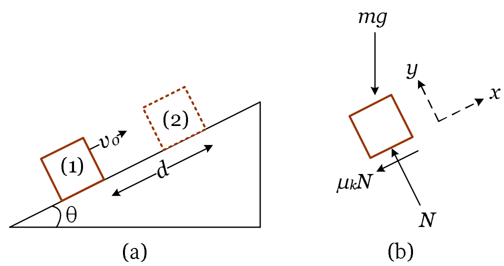

Fig. 3. Ramp surface. (a) schematic diagram, (b) free body diagram.

and the total energy of the pendulum at 2 is

$$U_2 = U_{2P} + U_{2K} = 0 + \left(\frac{1}{2}mv_{\text{max}}^2\right), \tag{7}$$

where subscripts P and K denote potential and kinetic energy, respectively. Symbol g_0 is the standard gravity (9.81 m/s^2), m is the mass of the pendulum, y_{max} is the maximum height of the pendulum, and v_{max} is the maximum speed of the pendulum. Equating (6) and (7), we obtain

$$\begin{aligned} U_1 &= U_2 \\ y_{\text{max}} &= \frac{1}{2}\frac{v_{\text{max}}^2}{g_0}, \end{aligned} \tag{8}$$

which gives a relationship between the maximum vertical displacement of the pendulum and its maximum speed. While this relationship was rather straightforward to derive, derivation of the same relationship using the Newton's law and kinematic equations is not trivial.

Consider another example: a box with mass m and initial speed v_0 placed on a ramp with inclination angle θ and kinetic friction coefficient μ_k, Fig. 3(a). In the presence of friction, the box stops after traveling a distance d. To find d using the energy method, note that the energy of the box at position 1 is equal to its kinetic energy,

$$U_1 = \frac{1}{2}mv_0^2. \tag{9}$$

This kinetic energy is converted to (i) potential energy at 2 because of the increase in the height of the box and (ii) heat because of the friction. We have

$$U_2 = (mg_0 \sin\theta)d + (mg_0 \cos\theta)\mu_k d. \tag{10}$$

From conservation of energy $U_1 = U_2$ and therefore

$$d = \frac{1}{2}\frac{v_0^2}{g_0(\sin\theta + \mu_k \cos\theta)}. \tag{11}$$

One could instead employ the Newton's second law of motion and kinematic equations to find the distance d. The forces acting on the box are illustrated in the free body diagram of Fig. 3(b). Using the second law in the x-direction, $ma_x = \sum F_x$, acceleration is found as

$$a_x = -g_0(sin\theta + \mu_k \cos\theta). \tag{12}$$

The box does not move in the y-direction.

Using the kinematic equation that relates the speed to acceleration and distance, $v^2 - v_0^2 = 2a_x d$, and realizing the box stops after traveling the distance d, we obtain the same expression as (11).

Note that while the results from both methods are identical, the energy method does not require decomposition of the variables in x- and y-directions; nor does it require an equation of motion. Therefore, the energy method offers a particularly convenient approach to solve this category of problems.

4 Energy Concept for Transient Stability Analysis

The equation governing the motion of the rotor of a synchronous generator is

$$J\frac{d^2\theta_m}{dt^2} + D\frac{d\theta_m}{dt} = T_a = T_m - T_e, \tag{13}$$

where J is the moment of inertia of the rotor, D is the damping constant, θ_m is the angular displacement of the rotor with respect to a stationary axis, T_a is the accelerating torque, T_m is the net mechanical torque on the shaft (input mechanical torque less losses), and T_e is the net electrical torque on the shaft (output electrical torque plus ri^2 losses). To simplify the equations, damping is neglected in the rest of this section. Since damping has a stabilizing effect, its omission leads to conservative results.

It is customary to replace θ_m with δ_m, where δ_m is measured with respect to a frame rotating at the synchronous speed w_{sm}. θ_m and δ_m are related as

$$\theta_m = \omega_{sm}t + \delta_m. \tag{14}$$

Therefore,

$$J\frac{d^2\delta_m}{dt^2} = T_m - T_e. \tag{15}$$

Multiply by ω_m

$$J\omega_m\frac{d^2\delta_m}{dt^2} = P_m - P_e. \tag{16}$$

The variation in the mechanical speed is negligible, and the product $J\omega_m$ is called the inertia constant M. Since M depends on the machine size, its value has wide variations. Therefore, a normalized quantity H is used instead, which is defined as

$$\begin{aligned} H &= \frac{\text{Kinetic energy of the rotor at } \omega_m}{\text{MVA}_{\text{rated}}} \\ &= \frac{1/2J\omega_{sm}^2}{S_{\text{rated}}}. \end{aligned} \tag{17}$$

Combining (16) and (17) and rearranging, we obtain

$$\frac{2H}{\omega_{sm}}\frac{d^2\delta_m}{dt^2} = \frac{P_m - P_e}{S_{\text{rated}}}, \tag{18}$$

which is called the *swing equation*.

The equal-area criterion is explained using the example power system of Fig. 4: a synchronous generator connected to an infinite bus through a double-circuit line and a step-up transformer. At time zero, a three-phase to ground fault occurs at point P on line 2. The fault changes the power flow in line 2, but line 1 continues to deliver power to the infinite bus. However, the reduction in the available ampacity of the lines may cause instability depending on (i) the parameters of the machine and network before and after the fault, e.g., impedance, transient reactance, the H constant, and level of loading, and (ii) the fault location and clearing time. The equal-area criterion provides a simple means to determine if the power system remains stable following this disturbance.

Fig. 5(a) shows the mechanical input power P_m and three power-angle curves for three conditions: the curve P_{pre} characterizes the pre-fault (normal) system, P_{fault} characterizes the faulted system, and P_{post} characterizes the system after the operation of the breakers (causing line 2 to go out of operation). These curves are obtained from the following expression, assuming the transmission lines are purely reactive:

$$P = \frac{E'V_\infty}{X}\sin\delta, \tag{19}$$

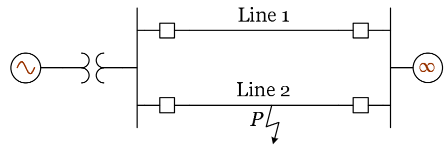

Fig. 4. One-line diagram of a synchronous generator connected to an infinite bus.

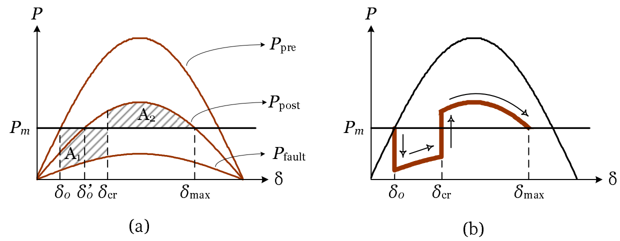

Fig. 5. Equal area criterion. (a) areas A_1 and A_2, (b) evolution of the machine angle.

where E' is the internal voltage of the generator, V_∞ is the voltage of the infinite bus, δ is the phase angle between E' and V_∞, and X is the total impedance between E' and V_∞, which includes the internal transient impedance of the synchronous generator, the impedance of the step-up transformer, and the equivalent impedance of the transmission lines. The equivalent impedance X has three distinct values in pre-, during, and post-fault intervals. At the steady-state angle δ_0, P_{pre} equals the mechanical input power P_m.

Upon occurrence of the fault, the power-angle characteristic of the system shifts to the P_{fault} curve, Fig. 5, causing the electric power P_e delivered to the infinite bus to decrease. From the swing equation,

$$\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2} = \frac{P_m - P_e}{S_{\text{rated}}}, \tag{20}$$

the accelerating power is positive, the machine speed increases beyond the synchronous speed, and as shown in Fig. 5(b), δ increases. The increase in the machine speed increases the kinetic energy of the machine. This energy is proportional to the area A_1, Fig. 5(a), between P_m and P_{fault} and between δ_0 and δ_{cr}. At δ_{cr}, the fault is cleared by tripping out line 2, and the power-angle characteristic of the system shifts to the P_{post} curve, Fig. 5(b). In this case, P_e is greater than P_m. As such, the machine decelerates, and its kinetic energy decreases. Since the machine speed is still higher than the synchronous speed, δ continues to increase. The maximum allowable increase in δ_m is δ_{max}; if δ increases beyond δ_{max}, the accelerating power $P_m - P_{\text{post}}$ becomes positive, causing the rotor to accelerate and the angle to increase further. This eventually leads to too high a deviation in the machine speed, and the machine will finally trip out.

The equal-area criterion states that for the power system to remain stable, the area A_1 must be less than the area A_2 (the area between P_m and P_{post} and between δ_{cr} and δ_{max}) [**?**]. That is, the increase in the rotor energy before clearing the fault must be equal to the decrease in the rotor energy after clearing the fault. In this case, the rotor swings between δ_{max} and δ_0. Due to mechanical losses, the oscillations gradually damp until the rotor settles at δ'_0, Fig. 5(a).

The critical clearing time (angle) is the maximum time (angle) of clearing a fault upon which the system can maintain its stability. This angle is found using the equal-area criterion $A_1 = A_2$. The areas A_1 and A_2

in Fig. 5(a) are obtained from

$$
\begin{aligned}
A_1 &= \int_{\delta_0}^{\delta_{\rm cr}} (P_m - P_{\rm fault}\sin\delta)\, d\delta \\
&= P_m(\delta_{\rm cr} - \delta_0) + P_{\rm fault}(\cos\delta_{\rm cr} - \cos\delta_0) \\
A_2 &= \int_{\delta_{\rm cr}}^{\delta_{\max}} (P_{\rm post}\sin\delta - P_m)\, d\delta \\
&= P_{\rm post}(\cos\delta_{\rm cr} - \cos\delta_{\max}) - P_m(\delta_{\max} - \delta_{\rm cr}).
\end{aligned} \tag{21}
$$

Equate A_1 and A_2 to find the critical clearing angle as

$$
\cos\delta_{\rm cr} = \frac{P_m(\delta_{\max} - \delta_0) - P_{\rm fault}\cos\delta_0 + P_{\rm post}\cos\delta_{\max}}{P_{\rm post} - P_{\rm fault}}. \tag{22}
$$