FDTD Dispersion Revisited: Faster-Than-Light Propagation

John B. Schneider, *Member, IEEE,* and Christopher L. Wagner

***Abstract*—The numerical dispersion relation that governs the propagation of fields in a finite-difference time-domain (FDTD) grid was derived several years ago. In this letter a different interpretation is given for the governing equation. It is shown that the dispersion relation predicts faster-than-light propagation for coarsely resolved fields. Additionally, some spectral components that were previously believed to have zero phase velocity are shown to propagate, albeit with exponential decay.**

***Index Terms*—FDTD methods.**

I. Introduction

TAFLOVE was the first to derive the dispersion relation for the second-order Yee finite-difference time-domain (FDTD) grid [1] (see also [2]). The formulas Taflove derived are correct, but the assumption that some coarsely resolved fields have a phase velocity of zero is not. As previously reported, the phase velocity decreases as the discretization becomes more coarse. However, in this letter it is shown that a threshold eventually is reached beyond which a further increase in the coarseness results in a wavenumber that is complex. For this complex wavenumber the phase velocity increases with grid coarseness and at no point is it zero. In fact, certain spectral components, which we refer to as superluminal, have a phase velocity greater than the speed of light. As shown below, these statements are straightforward to verify by comparison of predicted and measured FDTD fields.

We note that in most simulations fields are well resolved and one has little interest in the behavior of the spectral components whose wavenumbers are complex. Nevertheless, many FDTD researchers have observed superluminal fields and have wondered if they were a result of numeric noise (i.e., finite arithmetic) or the result of an inherent property of the FDTD grid. Here we demonstrate that superluminal propagation is indeed inherent to the grid and that it is accurately predicted by the familiar dispersion relation.

II. Dispersion Relation

To facilitate our reexamination of the dispersion relation, we restrict consideration to one dimension (which corresponds to grid-aligned propagation in higher dimensions), but similar arguments hold for nongrid-aligned propagation in two and three dimensions. An important distinction for the higher dimensions that is not considered here is that inhomogeneous plane waves, i.e., waves whose lines of constant amplitude are orthogonal to the lines of constant phase, are supported in higher dimensions. We restrict consideration to homogeneous plane waves in free space. However, for all dimensions the main point is that when searching for solutions to the dispersion relation, one should not restrict consideration to only real wavenumbers.

In one dimension, the dispersion relation is

$$\sin^2\left(\frac{\omega\Delta t}{2}\right) = \left(\frac{c\Delta t}{\delta}\right)^2 \sin^2\left(\frac{\tilde{k}\delta}{2}\right) \tag{1}$$

where

$\tilde{k}$	numeric wave number;
δ	spatial step size;
Δt	temporal step size;
ω	frequency.

We find it convenient to express the argument of the left-hand side sine function in terms of the points per wavelength and the Courant number

$$\frac{\omega\Delta t}{2} = \frac{2\pi}{2}\,\frac{c}{\lambda}\,\Delta t = \frac{\pi}{N_\lambda}\,\frac{c\Delta t}{\delta} = \frac{\pi}{N_\lambda}\,S$$

where $N_\lambda = \lambda/\delta$ is the points per wavelength and $S = c\Delta t/\delta$ is the Courant number (or stability factor). It is important to note that N_λ is defined in terms of the continuous-world wavelength λ and is *not* the number of points per wavelength of the field in the FDTD grid. Taking the square root of both sides of (1) and solving for $\tilde{k}\delta$ yields

$$\tilde{k}\delta = 2\,\sin^{-1}\left(\frac{1}{S}\,\sin\left(\frac{\pi}{N_\lambda}\,S\right)\right). \tag{2}$$

This equation governs propagation in one dimension but also describes propagation along the principal axes of the grid in two and three dimensions. When the Courant number is less than one, as it must be in two and three dimensions, the argument of the arc sine function can be greater than unity. When $S = \sin(S\pi/N_\lambda)$, the argument is unity and this is the threshold between wavenumbers that are real and ones that are complex. If N_λ is decreased such that the argument is greater than unity, there are no real values of $\tilde{k}$ that satisfy the dispersion relation. However, complex values of $\tilde{k}$ do permit a solution.

The complex wavenumbers that satisfy the dispersion relation for coarsely-resolved fields can be found as follows. The arc sine function is given by [3]

$$\sin^{-1}(\zeta) = -j\,\ln\left[j\zeta + \sqrt{1-\zeta^2}\right].$$

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The authors are with the School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752 USA.

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Assuming ζ is real and greater than one, this can be written

$$\sin^{-1}(\zeta) = \frac{\pi}{2} - j \ln\left[\zeta + \sqrt{\zeta^2 - 1}\right].$$

Using this in (2) yields

$$\tilde{k}\delta = \pi - j2 \ln\left[\zeta + \sqrt{\zeta^2 - 1}\right] \quad (3)$$

where $\zeta = \sin(S\pi/N_\lambda)/S$. For plane wave propagation given by $\exp(-j\tilde{k}\delta m)$ where m is the spatial index, the per-cell phase constant is π and the per-cell attenuation constant is $2 \ln\left[\zeta + \sqrt{\zeta^2 - 1}\right]$. Note that, provided ζ is greater than one, the phase constant is independent of the discretization.

Let us define a numeric phase velocity $\tilde{c} = \omega/\Re(\tilde{k})$ where $\Re(\cdot)$ indicates the real part. In the continuous world, the (exact) phase velocity is related to the wave number via $c = \omega/k$ and the continuous wave number in free space is given by $k = 2\pi/\lambda = 2\pi/(\delta N_\lambda)$. Taking the ratio of the numeric phase velocity to the exact phase velocity yields, upon canceling terms and multiplying numerator and denominator by δ, the following:

$$\frac{\tilde{c}}{c} = \frac{\omega/\Re(\tilde{k})}{\omega/k} = \frac{k\delta}{\Re(\tilde{k})\delta} = \frac{2\pi/N_\lambda}{\pi} = \frac{2}{N_\lambda}. \quad (4)$$

Thus, for frequencies so coarsely sampled that they have complex wave numbers, the numeric phase velocity is related to the exact speed of light by

$$\tilde{c} = \frac{2}{N_\lambda} c. \quad (5)$$

At this point one might wonder when there can be superluminal propagation since it seems that N_λ, the points per wavelength, should always be greater than two—that is, one would expect a discretization of at least two points per wavelength (otherwise the wave apparently would be below the spatial Nyquist rate), and hence the factor relating c and $\tilde{c}$ in (5) would always be less than one. In fact, N_λ can be less than two and all the spectral components corresponding to $N_\lambda < 2$ are superluminal. This seeming paradox is resolved by recalling that N_λ is defined in terms of the free-space wavelength and not the wavelength in the grid. Thus, even when a given frequency is such that the spatial sampling λ/δ is less than two, the energy corresponding to this spectral component is coupled into the grid (it is not spatially aliased into another frequency). The highest frequency that can be coupled into an FDTD grid is $f_{\max} = 1/(2\Delta t)$. The corresponding wavelength is $\lambda_{\min} = c/f_{\max} = 2c\Delta t$. Dividing this wavelength by the spatial step size yields the minimum value of N_λ which is $2c\Delta t/\delta = 2S$. Since in two and three dimensions the Courant number S is always less than one, the minimum value of N_λ is always less than two. Thus, the maximum numeric phase velocity is given by

$$\tilde{c}_{\max} = \frac{1}{S} c = \frac{\delta}{c\Delta t} c = \frac{\delta}{\Delta t}. \quad (6)$$

Note that $\tilde{c}_{\max}$ depends only on the temporal and spatial step sizes and is independent of the material parameters. Thus, the maximum speed of propagation is inherent to the grid itself and independent of the material present in the grid.

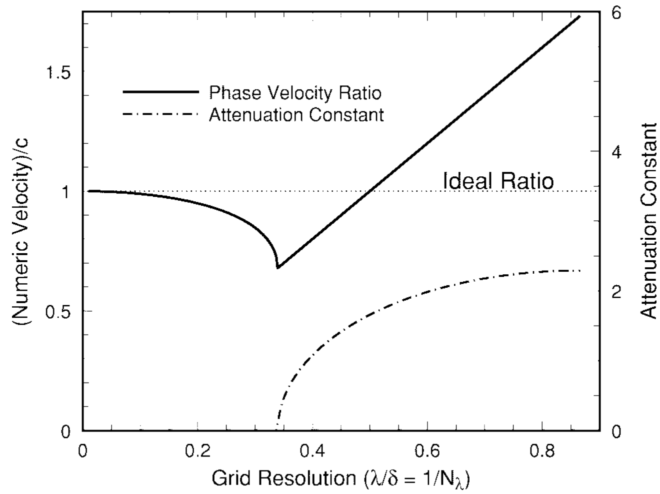

Fig. 1. Solid line is the ratio of the numeric phase velocity to true phase velocity versus grid resolution ($1/N_\lambda$). The Courant number is $1/\sqrt{3}$ which gives a minimum N_λ of $2/\sqrt{3} \approx 1.155$ or a maximum spatial step of 0.866λ. The dash-dot line is the per-cell attenuation constant versus grid resolution.

Fig. 1 shows the ratio of the numeric phase velocity and the exact phase velocity versus grid resolution (i.e., the size of the spatial step in fractions of a free-space wavelength). The Courant number corresponds to the three-dimensional (3-D) limit of $1/\sqrt{3}$ yielding a minimum N_λ of $2/\sqrt{3} \approx 1.155$ or a maximum spatial step size of approximately 0.866. The ideal ratio is unity indicated by the dotted horizontal line. Fig. 1 also shows the per-cell attenuation constant versus grid resolution using the same Courant number. The values of the attenuation constant are given on the right-hand side vertical scale. These plots show that as the wave number becomes complex, it first is subluminal and experiences exponential decay. As the spatial step size further increases, the field is superluminal, but it still experiences exponential decay.

III. Numerical Example

Superluminal propagation can be demonstrated easily. Perhaps more interesting, however, is that, given a full source description, the field at any point in a homogeneous grid can be accurately predicted via the dispersion relation. In other words, it is not necessary to perform an FDTD simulation to obtain the fields found at an arbitrary point in the grid. These fields can be predicted and they will "suffer" the same numeric dispersion that those in the FDTD grid experience. An obvious use for such a predictive ability would be in the construction of an "exact" total-field/scattered-field (TFSF) boundary that does not leak energy across the TFSF interface.

To demonstrate superluminal propagation, we construct a one-dimensional (1-D) computational domain with a "hard" Kronecker delta electric field source (the source is unity at the first time step and zero thereafter) located at the origin. The observation point is an electric field node ten cells away. The Courant number S is the Courant limit of a 3-D grid, i.e., $1/\sqrt{3}$ (demonstrating grid-aligned propagation in 3-D). In the physical world, it takes $10/S \approx 17.32$ time steps for light to travel from the source to the observation point. Thus, in this simulation, anything that arrives at, or before, the 17th time

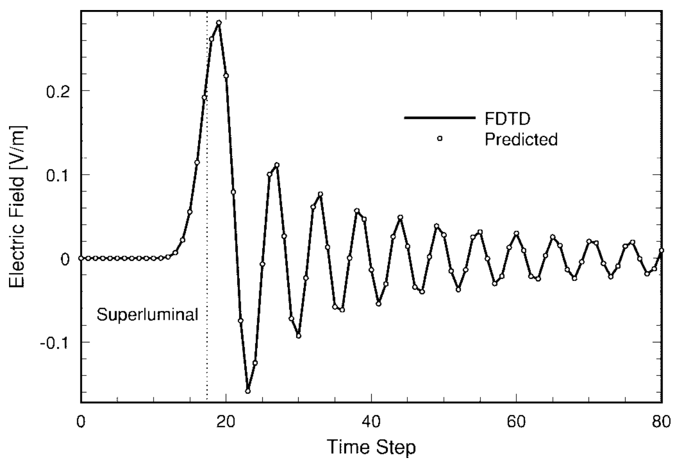

Fig. 2. Electric field ten cells away from a Kronecker delta source versus time. The boundary between super- and subluminal propagation is the dotted vertical line. The nonzero fields to the left of this line prove that the FDTD grid supports superluminal propagation.

step is superluminal. Fig. 2 shows (solid line) the field at the observation point versus time for an 80-time-step simulation. The superluminal region is to the left of the vertical dotted line. Clearly there are nonzero fields in this region thus proving that FDTD grids support superluminal propagation.

Next we demonstrate that the dispersion relation is indeed correct and from it the field at the observation point can be predicted. To do this, the spectrum of the source function is obtained; this is weighted by the grid transfer function (which is given by the dispersion relation), and the resulting function is inverse transformed back to the time domain. The source spectrum is given by

$$F(\omega) = \mathcal{F}(E_z(x=0, t)) \tag{7}$$

where $E_z(x=0, t)$ is the source node and $\mathcal{F}(\cdot)$ is the Fourier transform. The field as a function of time at a point N_x cells away from the source node is obtained via

$$E_z(x=N_x\delta, t) = \mathcal{F}^{-1}(\exp(-j\tilde{k}\delta N_x)F(\omega)) \tag{8}$$

where $\tilde{k}\delta$ is obtained from the dispersion relation (3).

Note that (7) and (8) are expressed in terms of continuous time and frequency. To solve these equations, however, we fall back on a discrete (numeric) solution, i.e., the transforms are done using FFT's. For the 80 time-step simulation described above, it is not sufficient to perform an 80-point DFT. Instead, one must use enough spectral samples to accurately resolve the entire k-space spectrum. To accomplish this, we used a 16 384-point FFT to obtain the predicted results. The Kronecker delta spectrum is white and represents a "worst case" scenario in that all the spatial wavenumbers are excited (this is in contrast to most realistic FDTD simulations where one would choose the source spectrum to be well resolved). Fig. 2 shows the predicted values as circles. Note that the predicted field corresponds precisely with the measured field in both the super- and subluminal regions. Even though the trailing edge of a pulse in an FDTD grid may appear as noise, the fluctuations are predictable given the dispersion relation.

It is possible to construct predicted fields that ignore the complex wavenumbers inherent in the dispersion relation. For example, if one sets to zero the transfer function for all superluminal frequencies, the predicted field will agree moderately well with the subluminal measured field. However, as expected, there is then no agreement with the superluminal field. Additionally, one can set to zero the transfer function for all spectral components that have complex wavenumbers. In this case, the measured and predicted fields can be made to agree moderately well if the observation point is far from the source point. This is because the fields with complex wavenumbers experience exponential decay as they propagate. Nevertheless, the agreement between the measured and predicted fields will be poor for observations points close to the source.

IV. Conclusion

The FDTD dispersion relation permits solutions with complex wavenumbers. Such complex waves can propagate faster than the speed of light. Furthermore, it is possible, using the dispersion relation, to predict fields in a homogeneous grid. Using these predicted fields, it may be possible to construct total-field/scattered field boundaries that do not leak energy. If the complex waves are ignored, the predicted field will not agree with the measured field.

Although it is desirable to avoid or minimize these complex waves, they will be present to some extent in virtually all FDTD simulations. Thus, these complex waves should be included in any complete analysis of such things as absorbing boundary conditions.

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