Chapter 10

Dispersive Material

10.1 Introduction

For many problems one can obtain acceptably accurate results by assuming material parameters are constants. However, constant material parameters are inherently an approximation. For example, it is impossible to have a lossless dielectric with constant permittivity (except, of course, for free space). If such a material did exist it would violate causality. (For a material to behave causally, the Kramers-Kronig relations show that for any deviation from free-space behavior the imaginary part of the permittivity or permeability, i.e., the loss, cannot vanish for all frequencies. Nevertheless, as far as causality is concerned, the loss can be arbitrarily small.)

A non-unity, scalar, constant relative permittivity is equivalent to assuming the polarization of charge within a material is instantaneous and in perfect proportion to the applied electric field. Furthermore, the reaction is the same at all frequencies, is the same in all directions, is the same for all times, and the same proportionality constant holds for all field strengths. In reality, essentially none of the these assumptions are absolutely correct. The relationship between the electric flux density $\mathbf{D}$ and the electric field $\mathbf{E}$ can reflect all the complexity of the real world. Instead of simply having $\mathbf{D} = \epsilon\mathbf{E}$ where ϵ is a scalar constant, one can make ϵ a tensor to describe different behaviors in different directions (off diagonal terms would indicate the amount of coupling from one direction to another). The permittivity can also be written as a nonlinear function of the applied electric field to account for nonlinear media. The material parameters can be functions of time (such as might pertain to a material which is being heated). Finally, one should not forget that the permittivity can be a function of position to account for spatial inhomogeneities.

When the speed of light in a material is a function of frequency, the material is said to be dispersive. The fact that the FDTD grid is dispersive has been discussed in Chap. 7. That dispersion is a numerical artifact and is distinct from the subject of this chapter. We have also considered lossy materials. Even when the conductivity of a material is assumed to be constant, the material is dispersive (ref. (5.69) which shows that the phase constant is not linearly proportional to the frequency which must be the case for non-dispersive propagation).

When the permittivity or permeability of a material are functions of frequency, the material is dispersive. In time-harmonic form one can account for the frequency dependence of permittivity by writing $\hat{\mathbf{D}}(\omega) = \hat{\epsilon}(\omega)\hat{\mathbf{E}}(\omega)$, where a caret is used to indicate a quantity in the frequency domain.

†Lecture notes by John Schneider. `fdtd-dispersive-material.tex`

This expression is simple in the frequency domain, but the FDTD method is a time-domain technique. The multiplication of harmonic functions is equivalent to convolution in the time domain. Therefore it requires some additional effort to model these types of dispersive materials. We start with a brief review of dispersive materials and then consider two ways in which to model such materials in the FDTD method.

10.2 Constitutive Relations and Dispersive Media

The electric flux density and magnetic field are related to the electric field and the magnetic flux density via

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \tag{10.1}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \tag{10.2}$$

where $\mathbf{P}$ and $\mathbf{M}$ account for the electric and magnetic dipoles, respectively, induced in the media. Keep in mind that force on a charge is a function of $\mathbf{E}$ and $\mathbf{B}$ so in some sense $\mathbf{E}$ and $\mathbf{B}$ are the "real" fields. The polarization vector $\mathbf{P}$ accounts for the local displacement of bound charge in a material. Because of the way in which $\mathbf{P}$ is constructed, by adding it to $\epsilon_0\mathbf{E}$ the resulting electric flux density $\mathbf{D}$ has the local effect of bound charge removed. In this way, the integral of $\mathbf{D}$ over a closed surface yields the free charge (thus Gauss's law, as expressed using the $\mathbf{D}$ field, is true whether material is present or not).

The magnetic field $\mathbf{H}$ ignores the local effect of bound charge in motion. Thus, the integration of $\mathbf{H}$ over a closed loop yields the current flowing through the surface enclosed by that loop where the current is due to either the flow of free charge or displacement current (i.e., the integral form of Ampere's law). Rearranging the terms in (10.2) and multiplying by μ_0 yields an expression for the magnetic flux density*, i.e.,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{10.3}$$

At a given frequency, for a linear, isotropic medium, the polarization vectors can be related to the electric and magnetic fields via an electric or magnetic susceptibility

$$\hat{\mathbf{P}}(\omega) = \epsilon_0\hat{\chi}_e(\omega)\hat{\mathbf{E}}(\omega), \tag{10.4}$$
$$\hat{\mathbf{M}}(\omega) = \hat{\chi}_m(\omega)\hat{\mathbf{H}}(\omega), \tag{10.5}$$

where $\hat{\chi}_e(\omega)$ and $\hat{\chi}_m(\omega)$ are the electric and magnetic susceptibility, respectively.† Thus we can write

$$\hat{\mathbf{D}}(\omega) = \epsilon_0\hat{\mathbf{E}}(\omega) + \epsilon_0\hat{\chi}_e(\omega)\hat{\mathbf{E}}(\omega), \tag{10.6}$$
$$\hat{\mathbf{B}}(\omega) = \mu_0\hat{\mathbf{H}}(\omega) + \mu_0\hat{\chi}_m(\omega)\hat{\mathbf{H}}(\omega). \tag{10.7}$$

*Note that there are those (e.g., Feynman) who advocate that one should avoid using $\mathbf{D}$ and $\mathbf{H}$. Others (e.g., Sommerfeld) have discussed the unfortunate naming of the magnetic field and the magnetic flux density. However, those issues are peripheral to the main subject of interest here and we will employ the notation and usage as is commonly found in engineering electromagnetics.

†Here we will assume that $\hat{\chi}_e(\omega)$ and $\hat{\chi}_m(\omega)$ are not functions of time. Thus frequency response of the material "today" is the same as it will be "tomorrow."

For the time being we restrict discussion to the electric fields where the permittivity $\hat{\epsilon}(\omega)$ is defined as

$$\hat{\epsilon}(\omega) = \epsilon_0 \hat{\epsilon}_r(\omega) = \epsilon_0(\epsilon_\infty + \hat{\chi}_e(\omega)). \tag{10.8}$$

where $\hat{\epsilon}_r$ is the relative permittivity and, as will be seen, the constant ϵ_∞ accounts for the effect of the charged material at high frequencies where the susceptibility function goes to zero.

The time-domain electric flux density can be obtain by inverse transforming (10.6). The product of $\hat{\chi}_e$ and $\hat{\mathbf{E}}$ in the frequency domain yields a convolution in the time domain. The fields are assumed to be zero prior to $t = 0$, so this yields

$$\mathbf{D}(t) = \epsilon_0 \mathbf{E}(t) + \int\limits_{\tau=0}^{t} \chi_e(\tau)\mathbf{E}(t-\tau)d\tau \tag{10.9}$$

where $\chi_e(t)$ is the inverse transform of $\hat{\chi}_e(\omega)$:

$$\chi_e(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{\chi}_e(\omega)e^{j\omega t}d\omega. \tag{10.10}$$

The time-domain function $\epsilon_0\chi_e(t)$ corresponds to the polarization vector $\mathbf{P}(t)$ for an impulsive electric field—effectively the impulse response of the medium.

We now consider three common susceptibility functions. As will be shown these are based on either simple mechanical or electrical models.

10.2.1 Drude Materials

In the Drude model, which often provides a good model of the behavior of conductors, charges are assumed to move under the influence of the electric field but they experience a damping force as well. This can be described by the following simple mechanical model

$$M\frac{d^2\mathbf{x}}{dt^2} = Q\mathbf{E}(t) - Mg\frac{d\mathbf{x}}{dt} \tag{10.11}$$

where M is the mass of the charge, g is the damping coefficient, Q is the amount of charge, and $\mathbf{x}$ is the displacement of the charge (the displacement is assumed to be in an arbitrary direction and not restricted to the x axis despite the use of the symbol $\mathbf{x}$). The left side of this equation is mass times acceleration and the right side is the sum of the forces on the charge, i.e., a driving force and a damping force. Rearranging this and converting to the frequency domain yields

$$M(j\omega)^2\hat{\mathbf{x}}(\omega) + Mg(j\omega)\hat{\mathbf{x}}(\omega) = Q\hat{\mathbf{E}}(\omega). \tag{10.12}$$

Thus the displacement can be expressed as

$$\hat{\mathbf{x}}(\omega) = -\frac{Q}{M\left(\omega^2 - jg\omega\right)}\hat{\mathbf{E}}(\omega). \tag{10.13}$$

The polarization vector $\mathbf{P}$ is related to the dipole moment of individual charges. If N is the number of dipoles per unit volume, the polarization vector is given by

$$\hat{\mathbf{P}} = NQ\hat{\mathbf{x}}. \tag{10.14}$$

Note that $\mathbf{P}$ has units of charge per units area (C/m^2) and thus, as would be expected from (10.1), has the same units as electric flux. Combining (10.13) and (10.14) yields

$$\hat{\mathbf{P}}(\omega) = -\epsilon_0 \frac{\frac{NQ^2}{M\epsilon_0}}{\omega^2 - jg\omega} \hat{\mathbf{E}}(\omega). \tag{10.15}$$

The electric susceptibility for Drude materials is thus given by

$$\hat{\chi}_e(\omega) = -\frac{\omega_p^2}{\omega^2 - jg\omega} \tag{10.16}$$

where $\omega_p^2 = NQ^2/(M\epsilon_0)$. However, we are not overly concerned with the specifics behind any one constant. For example, some authors may elect to combine the mass and damping coefficient which were kept as separate quantities in (10.11) (the product of the two dictates the damping force). Regardless of how the constants are defined, ultimately the Drude susceptibility will take the form shown in (10.16).

The relative permittivity for a Drude material can thus be written

$$\hat{\epsilon}_r(\omega) = \epsilon_\infty - \frac{\omega_p^2}{\omega^2 - jg\omega}. \tag{10.17}$$

Note that as ω goes to infinity the relative permittivity reduces to ϵ_∞. Consider a rather special case in which $\epsilon_\infty = 1$ and $g = 0$. When $\omega = \omega_p/\sqrt{2}$ the relative permittivity is -1. It is possible, at least to some extent, to construct a material which has not only this kind of behavior for permittivity but also for the behavior for the permeability, i.e., $\mu_r = \epsilon_r = -1$. This kind of material, which is known by various names including meta material, double-negative material, backward-wave material, and left-handed material, possesses many interesting properties. Some properties of these "meta materials" are also rather controversial as some people have made claims that others dispute (such as the ability to construct a "perfect" lens using a planar slab of this material).

The impulse response for the medium is the inverse Fourier transform of (10.16):

$$\chi_e(t) = \frac{\omega_p^2}{g}\left(1 - e^{-gt}\right) u(t) \tag{10.18}$$

where $u(t)$ is the unit step function. The factor g is seen to determine the rate at which the response goes to zero, i.e., its inverse is the relaxation time. The factor ω_p is known as the plasma frequency.

10.2.2 Lorentz Material

Lorentz material is based on a second-order mechanical model of charge motion. In this case, in addition to a damping force, there is a restoring force (effectively a spring force which wants to bring the charge back to its initial position). The sum of the forces can be expressed as

$$M\frac{d^2\mathbf{x}}{dt^2} = Q\mathbf{E}(t) - Mg\frac{d\mathbf{x}}{dt} - MK\mathbf{x}. \tag{10.19}$$

The terms in commons with those in (10.11) as they had in the case of the Drude model. The additional term represents the restoring force (which is proportional to the displacement) and is scaled by the spring constant K.

Converting to the frequency domain and rearranging yields

$$-M\omega^2\hat{\mathbf{x}}(\omega) + jMg\omega\hat{\mathbf{x}}(\omega) + MK\hat{\mathbf{x}}(\omega) = Q\hat{\mathbf{E}}(\omega). \tag{10.20}$$

Thus the displacement is given by

$$\hat{\mathbf{x}}(\omega) = \frac{Q}{M\left(K + jg\omega - \omega^2\right)}\hat{\mathbf{E}}(\omega). \tag{10.21}$$

Relating displacement to the polarization vector via $\hat{\mathbf{P}} = NQ\hat{\mathbf{x}}$ yields

$$\hat{\mathbf{P}}(\omega) = \epsilon_0 \frac{NQ^2}{M\epsilon_0\left(K + jg\omega - \omega^2\right)}\hat{\mathbf{E}}(\omega). \tag{10.22}$$

From this the susceptibility is identified as

$$\hat{\chi}_e(\omega) = \frac{NQ^2}{M\epsilon_0\left(K + jg\omega - \omega^2\right)} \tag{10.23}$$

However, as before, the important thing is the form of the function, not the individual constants. We thus write this as

$$\hat{\chi}_e(\omega) = \frac{\epsilon_\ell\omega_\ell^2}{\omega_\ell^2 + 2jg_\ell\omega - \omega^2} \tag{10.24}$$

where ω_ℓ is the undamped resonant frequency, g_ℓ is the damping coefficient, and ϵ_ℓ (together with ϵ_∞) accounts for the relatively permittivity at zero frequency. The corresponding relative permittivity is given by

$$\hat{\epsilon}_r(\omega) = \epsilon_\infty + \frac{\epsilon_\ell\omega_\ell^2}{\omega_\ell^2 + 2jg_\ell\omega - \omega^2}. \tag{10.25}$$

Note that when the frequency goes to zero the relative permittivity becomes $\epsilon_\infty + \epsilon_\ell$ whereas when the frequency goes to infinity the relative permittivity is simply ϵ_∞.

The time-domain form of the susceptibility function is given by the inverse transform of (10.24). This yields

$$\chi_e(t) = \frac{\epsilon_\ell\omega_\ell^2}{\sqrt{\omega_\ell^2 - g_\ell^2}} e^{-g_\ell t}\sin\left(t\sqrt{\omega_\ell^2 - g_\ell^2}\right)u(t). \tag{10.26}$$

10.2.3 Debye Material

Debye materials can be thought of as a simple RC circuit where the amount of polarization is related to the voltage across the capacitor. The "source" driving the circuit is the electric field. For a step in the source, there may be a constant polarization. As the frequency goes to infinity, the polarization goes to zero (leaving just the constant residual high-frequency term). The susceptibility is thus written

$$\hat{\chi}_e(\omega) = \frac{\epsilon_d}{1 + j\omega\tau_d} \tag{10.27}$$

where τ_d is the time constant and ϵ_d (together with ϵ_∞) accounts for the relative permittivity when the frequency is zero. The relative permittivity is given by

$$\hat{\epsilon}_r(\omega) = \epsilon_\infty + \frac{\epsilon_d}{1 + j\omega\tau_d}. \tag{10.28}$$

The time-domain form of the susceptibility function is

$$\chi_e(t) = \frac{\epsilon_d}{\tau_d} e^{-t/\tau_d} u(t). \tag{10.29}$$

10.3 Debye Materials Using the ADE Method

The finite-difference approximation of the differential equation that relates the polarization and the electric field can be used to obtain the polarization at future times in terms of its past value and an expression involving the electric field. By doing so, one can obtain a consistent FDTD model that requires that a quantity related to the polarization be stored as an additional variable. This approach is known as the auxiliary differential equation (ADE) method.

In the frequency domain Ampere's law can be written

$$\epsilon_0 \epsilon_\infty j\omega \hat{\mathbf{E}} + \sigma \hat{\mathbf{E}} + \hat{\mathbf{J}}_p = \nabla \times \hat{\mathbf{H}} \tag{10.30}$$

where the polarization current $\hat{\mathbf{J}}_p$ is given by

$$\hat{\mathbf{J}}_p = j\omega \hat{\mathbf{P}} = j\omega \epsilon_0 \hat{\chi}_e \hat{\mathbf{E}}. \tag{10.31}$$

For a Debye material this becomes

$$\hat{\mathbf{J}}_p = j\omega \epsilon_0 \frac{\epsilon_d}{1 + j\omega\tau} \hat{\mathbf{E}}. \tag{10.32}$$

Multiplying through by $1 + j\omega\tau$ yields

$$\hat{\mathbf{J}}_p + j\omega\tau \hat{\mathbf{J}}_p = j\omega \epsilon_0 \epsilon_d \hat{\mathbf{E}}. \tag{10.33}$$

Converting to the time domain produces

$$\mathbf{J}_p + \tau \frac{\partial \mathbf{J}_p}{\partial t} = \epsilon_0 \epsilon_d \frac{\partial \mathbf{E}}{\partial t}. \tag{10.34}$$

Discretizing this about the time-step $q + 1/2$ yields

$$\frac{\mathbf{J}_p^{q+1} + \mathbf{J}_p^q}{2} + \tau \frac{\mathbf{J}_p^{q+1} - \mathbf{J}_p^q}{\Delta_t} = \epsilon_0 \epsilon_d \frac{\mathbf{E}^{q+1} - \mathbf{E}^q}{\Delta_t}. \tag{10.35}$$

Since we are assuming an isotropic medium, the polarization current is aligned with the electric field. Hence in (10.35) the x component of $\mathbf{J}_p$ depends only on the x component of $\mathbf{E}$ (and similarly for the y and z components).

Solving (10.35) for $\mathbf{J}_p^{q+1}$ yields

$$\mathbf{J}_p^{q+1} = \frac{1 - \frac{\Delta_t}{2\tau}}{1 + \frac{\Delta_t}{2\tau}} \mathbf{J}_p^q + \frac{\frac{\Delta_t}{\tau}}{1 + \frac{\Delta_t}{2\tau}} \frac{\epsilon_0 \epsilon_d}{\Delta_t} \left(\mathbf{E}^{q+1} - \mathbf{E}^q \right). \tag{10.36}$$

Whatever the time-constant τ is, it can be expressed in terms of some multiple of the time-step, i.e., $\tau = N_\tau \Delta_t$ where N_τ does not need to be an integer. As will be shown, it is convenient to multiply both sides of (10.36) by the spatial step size. We will assume a uniform grid in which $\Delta_x = \Delta_y = \Delta_z = \delta$. Thus (10.36) can be written

$$\delta \mathbf{J}_p^{q+1} = \frac{1 - \frac{1}{2N_\tau}}{1 + \frac{1}{2N_\tau}} \delta \mathbf{J}_p^q + \frac{\frac{1}{N_\tau}}{1 + \frac{1}{2N_\tau}} \frac{\epsilon_0 \epsilon_d \delta}{\Delta_t} \left(\mathbf{E}^{q+1} - \mathbf{E}^q \right). \tag{10.37}$$

Consider the factor $\epsilon_0 \epsilon_d \delta / \Delta_t$:

$$\frac{\epsilon_0 \epsilon_d \delta}{\Delta_t} = \frac{\sqrt{\epsilon_0 \mu_0} \epsilon_d \delta}{\sqrt{\frac{\mu_0}{\epsilon_0}} \Delta_t} = \frac{\epsilon_d \delta}{\eta_0 c \Delta_t} = \frac{\epsilon_d}{\eta_0 S_c}. \tag{10.38}$$

Therefore (10.37) can be written

$$\delta \mathbf{J}_p^{q+1} = C_{jj} \delta \mathbf{J}_p^q + C_{je} \left(\mathbf{E}^{q+1} - \mathbf{E}^q \right), \tag{10.39}$$

where

$$C_{jj} = \frac{1 - \frac{1}{2N_\tau}}{1 + \frac{1}{2N_\tau}}, \tag{10.40}$$

$$C_{je} = \frac{\frac{1}{N_\tau}}{1 + \frac{1}{2N_\tau}} \frac{\epsilon_d}{\eta_0 S_c}. \tag{10.41}$$

Recall that Ampere's law is discretized about the time-step $(q + 1/2)\Delta_t$. Since the polarization current appears in Ampere's law, we thus need an expression for $\delta \mathbf{J}_p^{q+1/2}$. This is simply given by the average of $\delta \mathbf{J}_p^{q+1}$ (which is given by (10.39)) and $\delta \mathbf{J}_p^q$:

$$\delta \mathbf{J}_p^{q+1/2} = \frac{\delta \mathbf{J}_p^{q+1} + \delta \mathbf{J}_p^q}{2} = \frac{1}{2} \left([1 + C_{jj}] \delta \mathbf{J}_p^q + C_{je} \left(\mathbf{E}^{q+1} - \mathbf{E}^q \right) \right). \tag{10.42}$$

The discrete time-domain form of Ampere's law expanded about the time-step $(q + 1/2)$ is

$$\epsilon_0 \epsilon_\infty \frac{\mathbf{E}^{q+1} - \mathbf{E}^q}{\Delta_t} + \sigma \frac{\mathbf{E}^{q+1} + \mathbf{E}^q}{2} + \mathbf{J}_p^{q+1/2} = \nabla \times \mathbf{H}^{q+1/2}. \tag{10.43}$$

Multiplying through by δ and using (10.42) in (10.43) yields

$$\frac{\epsilon_\infty \epsilon_0 \delta}{\Delta_t} \left(\mathbf{E}^{q+1} - \mathbf{E}^q \right) + \frac{\sigma \delta}{2} \left(\mathbf{E}^{q+1} + \mathbf{E}^q \right) + \frac{1}{2} \left([1 + C_{jj}] \delta \mathbf{J}_p^q + C_{je} \left(\mathbf{E}^{q+1} - \mathbf{E}^q \right) \right) = \delta \nabla \times \mathbf{H}^{q+1/2}. \tag{10.44}$$

The curl of the magnetic field, which involves spatial derivatives, will have a δ in the denominator of the finite differences. Thus $\delta\nabla \times \mathbf{H}^{q+1/2}$ will involve merely the difference of the various magnetic-field components.

Regrouping terms in (10.44) produces

$$\mathbf{E}^{q+1}\left(\frac{\epsilon_\infty\epsilon_0\delta}{\Delta_t}+\frac{\sigma\delta}{2}+\frac{1}{2}C_{je}\right)=\mathbf{E}^{q}\left(\frac{\epsilon_\infty\epsilon_0\delta}{\Delta_t}-\frac{\sigma\delta}{2}+\frac{1}{2}C_{je}\right)+\delta\nabla\times\mathbf{H}^{q+1/2}-\frac{1}{2}\left[1+C_{jj}\right]\delta\mathbf{J}_p^q. \tag{10.45}$$

The term multiplying the electric fields can be written as

$$\frac{\epsilon_\infty\epsilon_0\delta}{\Delta_t}\left(1\pm\frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0}+\frac{C_{je}\Delta_t}{2\epsilon_\infty\epsilon_0\delta}\right)=\frac{\epsilon_\infty}{\eta_0 S_c}\left(1\pm\frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0}+\frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}\right). \tag{10.46}$$

Therefore (10.45) can be written

$$\mathbf{E}^{q+1}=\frac{1-\frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0}+\frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}}{1+\frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0}+\frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}}\mathbf{E}^q+\frac{\frac{\eta_0 S_c}{\epsilon_\infty}}{1+\frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0}+\frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}}\left(\delta\nabla\times\mathbf{H}^{q+1/2}-\frac{1}{2}\left[1+C_{jj}\right]\delta\mathbf{J}_p^q\right). \tag{10.47}$$

The way in which the term $\sigma\Delta_t/(2\epsilon_0)$ could be expressed in terms of the skin depth was discussed in Sec. 5.7.

The FDTD model of a Debye material would be implemented as follows:

1. Update the magnetic fields in the usual way. $\mathbf{H}^{q-1/2} \Rightarrow \mathbf{H}^{q+1/2}$.
2. For each electric-field component, do the following:
   (a) Copy the electric field to a temporary variable. $E_{tmp} = E^q$.
   (b) Update the electric field using (10.47). $E^q \Rightarrow E^{q+1}$.
   (c) Update the polarization current (actually δ times the polarization current) using (10.39). $J_p^q \Rightarrow J_p^{q+1}$. This update requires both E^{q+1} and E^q (which is stored in E_{tmp}).
3. Repeat.

The polarization current (actually the product of the spatial step-size and the polarization current, i.e., $\delta\mathbf{J}_p$) must be stored as a separate quantity. Of course it only needs to be stored for nodes at which it is non-zero.

If the polarization current is initially zero and C_{je} is zero, the polarization current will always be zero and the material behaves as a standard non-dispersive material (although, of course, dispersion is always present in the grid itself). Thus the update equations presented here can be used throughout a simulation which is a mix of dispersive and non-dispersive media. One merely has to set the constants to the appropriate values for a given location.

10.4 Drude Materials Using the ADE Method

The electric susceptibility for a Drude material is given in (10.16) and can also be written

$$\hat{\chi}_e(\omega)=\frac{\omega_p^2}{j\omega(j\omega+g)}. \tag{10.48}$$

The associated polarization current is

$$\hat{\mathbf{J}}_p = j\omega\hat{\mathbf{P}} = j\omega\epsilon_0\hat{\chi}_e\hat{\mathbf{E}} = j\omega\epsilon_0\frac{\omega_p^2}{j\omega(j\omega+g)}\hat{\mathbf{E}}. \tag{10.49}$$

Canceling $j\omega$ and cross multiplying by $g + j\omega$ yields

$$g\hat{\mathbf{J}}_p + j\omega\hat{\mathbf{J}}_p = \epsilon_0\omega_p^2\hat{\mathbf{E}}. \tag{10.50}$$

Expressed in the time-domain, this is

$$g\mathbf{J}_p + \frac{\partial\mathbf{J}_p}{\partial t} = \epsilon_0\omega_p^2\mathbf{E}. \tag{10.51}$$

Discretizing time and expanding this about the time-step $q + 1/2$ yields

$$g\frac{\mathbf{J}_p^{q+1} + \mathbf{J}_p^q}{2} + \frac{\mathbf{J}_p^{q+1} - \mathbf{J}_p^q}{\Delta_t} = \epsilon_0\omega_p^2\frac{\mathbf{E}^{q+1} + \mathbf{E}^q}{2}. \tag{10.52}$$

Solving for $\mathbf{J}_p^{q+1}$ we obtain

$$\mathbf{J}_p^{q+1} = \frac{1 - \frac{g\Delta_t}{2}}{1 + \frac{g\Delta_t}{2}}\mathbf{J}_p^q + \frac{1}{1 + \frac{g\Delta_t}{2}}\frac{\epsilon_0\omega_p^2\Delta_t}{2}\left(\mathbf{E}^{q+1} + \mathbf{E}^q\right). \tag{10.53}$$

The damping or loss term g is the inverse of the relaxation time and hence can be expressed as a multiple of the number of time steps, i.e.,

$$g = \frac{1}{N_g\Delta_t} \tag{10.54}$$

where N_g does not need to be integer. Consider the term multiplying the electric field

$$\frac{\epsilon_0\omega_p^2\Delta_t}{2} = \frac{\sqrt{\epsilon_0\mu_0}4\pi^2 f_p^2\Delta_t}{\sqrt{\frac{\mu_0}{\epsilon_0}}2} = \frac{2\pi^2c^2\Delta_t}{\eta_0\lambda_p^2 c} = \frac{2\pi^2 c\Delta_t}{\eta_0(N_p\delta)^2} = \frac{2\pi^2 S_c}{\eta_0 N_p^2\delta} \tag{10.55}$$

where f_p is the plasma frequency in Hertz, λ_p is the free-space wavelength at this frequency, and N_p is the number of points per wavelength at the plasma frequency. Since this term contains δ in the denominator, we multiply (10.53) by δ to obtain

$$\delta\mathbf{J}_p^{q+1} = C_{jj}\delta\mathbf{J}_p^q + C_{je}\left(\mathbf{E}^{q+1} + \mathbf{E}^q\right) \tag{10.56}$$

where

$$C_{jj} = \frac{1 - \frac{1}{2N_g}}{1 + \frac{1}{2N_g}}, \tag{10.57}$$

$$C_{je} = \frac{1}{1 + \frac{1}{2N_g}}\frac{2\pi^2 S_c}{\eta_0 N_p^2}. \tag{10.58}$$

Note the similarity between (10.39) and (10.56). These equations have nearly identical forms but the constants are different and there is a different sign associated with the "old" value of the electric field.

The general discretized form of Ampere's law expanded about the time-step $(q + 1/2)$ is unchanged from (10.43) and is repeat below

$$\epsilon_0\epsilon_\infty \frac{\mathbf{E}^{q+1} - \mathbf{E}^q}{\Delta_t} + \sigma \frac{\mathbf{E}^{q+1} + \mathbf{E}^q}{2} + \mathbf{J}_p^{q+1/2} = \nabla \times \mathbf{H}^{q+1/2}. \tag{10.59}$$

As before, $\delta\mathbf{J}_p^{q+1/2}$ can be obtained by the average of $\delta\mathbf{J}_p^{q+1}$ and $\delta\mathbf{J}_p^q$:

$$\delta\mathbf{J}_p^{q+1/2} = \frac{\delta\mathbf{J}_p^{q+1} + \delta\mathbf{J}_p^q}{2} = \frac{1}{2}\left([1 + C_{jj}]\delta\mathbf{J}_p^q + C_{je}\left(\mathbf{E}^{q+1} + \mathbf{E}^q\right)\right). \tag{10.60}$$

Multiplying through by δ and using (10.60) for the polarization current yields

$$\frac{\epsilon_\infty\epsilon_0\delta}{\Delta_t}\left(\mathbf{E}^{q+1} - \mathbf{E}^q\right) + \frac{\sigma\delta}{2}\left(\mathbf{E}^{q+1} + \mathbf{E}^q\right) + \frac{1}{2}\left([1 + C_{jj}]\delta\mathbf{J}_p^q + C_{je}\left(\mathbf{E}^{q+1} + \mathbf{E}^q\right)\right) = \delta\nabla \times \mathbf{H}^{q+1/2}. \tag{10.61}$$

Note that, other than the constants being those for a Drude material, the only way in which this expression differs from (10.44) is the sign of the "old" electric field associated with the polarization current. Therefore (10.47) again yields the expression for the "future" electric field if we make the appropriate change of sign. The result is

$$\mathbf{E}^{q+1} = \frac{1 - \frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0} - \frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}}{1 + \frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0} + \frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}}\mathbf{E}^q + \frac{\frac{\eta_0 S_c}{\epsilon_\infty}}{1 + \frac{\sigma\Delta_t}{2\epsilon_\infty\epsilon_0} + \frac{C_{je}\eta_0 S_c}{2\epsilon_\infty}}\left(\delta\nabla \times \mathbf{H}^{q+1/2} - \frac{1}{2}\left[1 + C_{jj}\right]\delta\mathbf{J}_p^q\right). \tag{10.62}$$

The implementation of an FDTD algorithm for Drude material now parallels that for Debye material:

1. Update the magnetic fields in the usual way. $\mathbf{H}^{q-1/2} \Rightarrow \mathbf{H}^{q+1/2}$.

2. For each electric-field component, do the following

   (a) Copy the electric field to a temporary variable. $E_{tmp} = E^q$.

   (b) Update the electric field using (10.62). $E^q \Rightarrow E^{q+1}$.

   (c) Update the polarization current (actually δ times the polarization current) using (10.56). $J^q \Rightarrow J^{q+1}$. This update requires both E^{q+1} and E^q (which is stored in E_{tmp}).

3. Repeat.

10.5 Magnetically Dispersive Material

In the frequency domain Faraday's law is

$$\nabla \times \hat{\mathbf{E}} = -j\omega\hat{\mathbf{B}} - \sigma_m\hat{\mathbf{H}} \tag{10.63}$$

where σ_m is the magnetic conductivity. Generalizing (10.7) slightly, the permeability can be written as

$$\hat{\mu}(\omega) = \mu_0 \hat{\mu}_r(\omega) = \mu_0(\mu_\infty + \hat{\chi}_m(\omega)). \tag{10.64}$$

where the factor μ_∞ accounts for the permeability at high frequencies. Faraday's law can thus be written

$$\nabla \times \hat{\mathbf{E}} = -j\omega\mu_0\mu_\infty \hat{\mathbf{H}}(\omega) - \sigma_m \hat{\mathbf{H}} - \hat{\mathbf{J}}_m. \tag{10.65}$$

where the magnetic polarization current $\hat{\mathbf{J}}_m$ is given by

$$\hat{\mathbf{J}}_m = j\omega\mu_0 \hat{\mathbf{M}} = j\omega\mu_0 \hat{\chi}_m(\omega)\hat{\mathbf{H}}(\omega). \tag{10.66}$$

The derivation of the equations which govern an FDTD implementation of a magnetically dispersive material parallel that of electrically dispersive material. Here we consider the case of Drude dispersion where

$$\hat{\chi}_m(\omega) = -\frac{\omega_p^2}{\omega^2 - jg\omega} = \frac{\omega_p^2}{j\omega(j\omega + g)}. \tag{10.67}$$

(The plasma frequency ω_p and damping term g for magnetic susceptibility are distinct from those of electric susceptibility.) Thus the polarization current and magnetic field are related by

$$\hat{\mathbf{J}}_m = j\omega\mu_0 \frac{\omega_p^2}{j\omega(j\omega + g)}\hat{\mathbf{H}} = \frac{\mu_0\omega_p^2}{(j\omega + g)}\hat{\mathbf{H}}. \tag{10.68}$$

Multiplying by $g + j\omega$ yields

$$(g + j\omega)\hat{\mathbf{J}}_m = \mu_0\omega_p^2 \hat{\mathbf{H}}. \tag{10.69}$$

The time-domain equivalent of this is

$$g\mathbf{J}_m + \frac{\partial \mathbf{J}_m}{\partial t} = \mu_0\omega_p^2 \mathbf{H}. \tag{10.70}$$

Discretizing time and expanding this about the time-step q yields

$$g\frac{\mathbf{J}_m^{q+1/2} + \mathbf{J}_m^{q-1/2}}{2} + \frac{\mathbf{J}_m^{q+1/2} - \mathbf{J}_m^{q-1/2}}{\Delta_t} = \mu_0\omega_p^2 \frac{\mathbf{H}^{q+1/2} + \mathbf{H}^{q-1/2}}{2}. \tag{10.71}$$

Solving for $\mathbf{J}_m^{q+1/2}$ yields

$$\mathbf{J}_m^{q+1/2} = \frac{1 - \frac{g\Delta_t}{2}}{1 + \frac{g\Delta_t}{2}}\mathbf{J}_m^{q-1/2} + \frac{1}{1 + \frac{g\Delta_t}{2}} \frac{\mu_0\omega_p^2\Delta_t}{2}\left(\mathbf{H}^{q+1/2} + \mathbf{H}^{q-1/2}\right). \tag{10.72}$$

Consider the term multiplying the magnetic field

$$\frac{\mu_0\omega_p^2\Delta_t}{2} = \frac{\sqrt{\frac{\mu_0}{\epsilon_0}}\sqrt{\epsilon_0\mu_0}4\pi^2 f_p^2\Delta_t}{2} = \frac{2\pi^2\eta_0 c^2\Delta_t}{\lambda_p^2 c} = \frac{2\pi^2\eta_0 c\Delta_t}{(N_p\delta)^2} = \frac{2\pi^2\eta_0 S_c}{N_p^2\delta}. \tag{10.73}$$

where, as before, f_p is the plasma frequency in Hertz, λ_p is the free-space wavelength at this frequency, and N_p is the number of points per wavelength at the plasma frequency. Again expressing the damping coefficient in terms of some multiple of the time step, i.e., $g = 1/(N_g\Delta_t)$, multiplying all terms in (10.72) by δ, and employing the final form of the term given in (10.73), the update equation for the polarization current can be written as

$$\delta\mathbf{J}_m^{q+1/2} = C_{jj}\delta\mathbf{J}_m^{q-1/2} + C_{jh}\left(\mathbf{H}^{q+1/2} + \mathbf{H}^{q-1/2}\right) \tag{10.74}$$

where

$$C_{jj} = \frac{1 - \frac{1}{2N_g}}{1 + \frac{1}{2N_g}}, \tag{10.75}$$

$$C_{jh} = \frac{1}{1 + \frac{1}{2N_g}}\frac{2\pi^2\eta_0 S_c}{N_p^2}. \tag{10.76}$$

To obtain the magnetic polarization current at time-step q, the current at time-steps $q + 1/2$ and $q - 1/2$ are averaged:

$$\delta\mathbf{J}_m^q = \frac{\delta\mathbf{J}_m^{q+1/2} + \delta\mathbf{J}_m^{q-1/2}}{2} = \frac{1}{2}\left([1 + C_{jj}]\delta\mathbf{J}_m^{q-1/2} + C_{jh}\left(\mathbf{H}^{q+1/2} + \mathbf{H}^{q-1/2}\right)\right). \tag{10.77}$$

The discretized form of Faraday's law expanded about time-step q is

$$-\mu_0\mu_\infty\frac{\mathbf{H}^{q+1/2} - \mathbf{H}^{q-1/2}}{\Delta_t} - \sigma_m\frac{\mathbf{H}^{q+1/2} - \mathbf{H}^{q-1/2}}{2} - \mathbf{J}_m^q = \nabla \times \mathbf{E}^q. \tag{10.78}$$

Multiplying through by $-\delta$ and using (10.77) for the polarization current yields

$$\begin{aligned}\frac{\mu_\infty\mu_0\delta}{\Delta_t}\left(\mathbf{H}^{q+1/2} - \mathbf{H}^{q-1/2}\right) + \frac{\sigma_m\delta}{2}\left(\mathbf{H}^{q+1/2} + \mathbf{H}^{q-1/2}\right)& \\ + \frac{1}{2}\left([1 + C_{jj}]\delta\mathbf{J}_m^{q-1/2} + C_{jh}\left(\mathbf{H}^{q+1/2} + \mathbf{H}^{q-1/2}\right)\right) &= -\delta\nabla \times \mathbf{E}^q.\end{aligned} \tag{10.79}$$

After regrouping terms we obtain

$$\begin{aligned}\mathbf{H}^{q+1/2}\left(\frac{\mu_\infty\mu_0\delta}{\Delta_t} + \frac{\sigma_m\delta}{2} + \frac{1}{2}C_{jh}\right) =& \\ \mathbf{H}^{q-1/2}\left(\frac{\mu_\infty\mu_0\delta}{\Delta_t} - \frac{\sigma_m\delta}{2} + \frac{1}{2}C_{jh}\right) - \delta\nabla \times \mathbf{E}^q - \frac{1}{2}\left[1 + C_{jj}\right]\delta\mathbf{J}_m^{q-1/2}.&\end{aligned} \tag{10.80}$$

The factor $\mu_0\delta/\Delta_t$ is equivalent to η_0/S_c so that the update equation for the magnetic field can be written

$$\mathbf{H}^{q+1/2} = \frac{1 - \frac{\sigma_m\Delta_t}{2\mu_\infty\mu_0} - \frac{C_{jh}S_c}{2\eta_0\mu_\infty}}{1 + \frac{\sigma_m\Delta_t}{2\mu_\infty\mu_0} + \frac{C_{jh}S_c}{2\eta_0\mu_\infty}}\mathbf{H}^{q-1/2} + \frac{\frac{S_c}{\eta_0\mu_\infty}}{1 + \frac{\sigma_m\Delta_t}{2\mu_\infty\mu_0} + \frac{C_{jh}S_c}{2\eta_0\mu_\infty}}\left(-\delta\nabla \times \mathbf{E}^q - \frac{1}{2}\left[1 + C_{jj}\right]\delta\mathbf{J}_m^{q-1/2}\right). \tag{10.81}$$

An FDTD model of a magnetically dispersive material would be implemented as follows:

1. For each magnetic-field component, do the following

(a) Copy the magnetic field to a temporary variable. $H_{tmp} = H^{q-1/2}$.

(b) Update the magnetic field using (10.81). $H^{q-1/2} \Rightarrow H^{q+1/2}$.

(c) Update the polarization current (actually δ times the polarization current) using (10.74). $J_m^{q-1/2} \Rightarrow J_m^{q+1/2}$. This update requires both $H^{q+1/2}$ and $H^{q-1/2}$ (which is stored in H_{tmp}).

2. Update the electric fields in whatever way is appropriate (which may include the dispersive implementations described previously) $\mathbf{E}^q \Rightarrow \mathbf{E}^{q+1}$.

3. Repeat.

10.6 Piecewise Linear Recursive Convolution

An alternative implementation of dispersive material is offered by the piecewise linear recursive convolution (PLRC) method. Recall that multiplication in the frequency domain is equivalent to convolution in the time domain. Thus, the frequency-domain relationship

$$\hat{\mathbf{D}}(\omega) = \epsilon_0\epsilon_\infty\hat{\mathbf{E}}(\omega) + \epsilon_0\hat{\chi}_e(\omega)\hat{\mathbf{E}}(\omega) \tag{10.82}$$

is equivalent to the time-domain relationship

$$\mathbf{D}(t) = \epsilon_0\epsilon_\infty\mathbf{E}(t) + \epsilon_0\int\limits_{\zeta=0}^{t}\mathbf{E}(t-\zeta)\chi_e(\zeta)d\zeta \tag{10.83}$$

where ζ is a dummy variable of integration. In discrete form the electric flux density at time-step $q\Delta_t$ can be written

$$\mathbf{D}^q = \epsilon_0\epsilon_\infty\mathbf{E}^q + \epsilon_0\int\limits_{\zeta=0}^{q\Delta_t}\mathbf{E}(q\Delta_t-\zeta)\chi_e(\zeta)d\zeta. \tag{10.84}$$

Although in an FDTD simulation the electric field $\mathbf{E}$ would only be available at discrete points in time, we wish to treat the field as if it varies continuously insofar as the integral is concerned. This is accomplished by assuming the field varies linearly between sample points. For example, assume the continuous variable t is between $i\Delta_t$ and $(i+1)\Delta_t$. Over this range the electric field is approximated by

$$\mathbf{E}(t) = \mathbf{E}^i + \frac{\mathbf{E}^{i+1}-\mathbf{E}^i}{\Delta_t}(t-i\Delta_t) \quad \text{for} \quad i\Delta_t \le t \le (i+1)\Delta_t. \tag{10.85}$$

When t is equal to $i\Delta_t$ we obtain $\mathbf{E}^i$ and when t is $(i+1)\Delta_t$ we obtain $\mathbf{E}^{i+1}$. The field varies linearly between these points.

To obtain a more general representation of the electric field, let us define the pulse function $p_i(t)$ which is given by

$$p_i(t) = \begin{cases} 1 & \text{if} \quad i\Delta_t \le t < (i+1)\Delta_t, \\ 0 & \text{otherwise.} \end{cases} \tag{10.86}$$

Using this pulse function the electric field can be written as

$$\mathbf{E}(t) = \sum_{i=0}^{M-1} \left[\mathbf{E}^i + \frac{\mathbf{E}^{i+1} - \mathbf{E}^i}{\Delta_t}(t - i\Delta_t) \right] p_i(t). \tag{10.87}$$

This provides a piecewise-linear approximation of the electric field over M segments. Despite the summation, the pulse function ensures that only one segment is turned on for any given value of t. Hence the summation can be thought of as serving more to collect together the various segments rather than as serving to add several terms.

In the integrand of (10.84) the argument of the electric field is $q\Delta_t - \zeta$ where $q\Delta_t$ is constant with respect to the variable of integration ζ. When ζ varies from $i\Delta_t$ to $(i+1)\Delta_t$ the electric field should vary from the discrete points $\mathbf{E}^{q-i}$ to $\mathbf{E}^{q-i-1}$. Thus, the electric field can be represented by

$$\mathbf{E}(q\Delta_t - \zeta) = \mathbf{E}^{q-i} + \frac{\mathbf{E}^{q-i-1} - \mathbf{E}^{q-i}}{\Delta_t}(\zeta - i\Delta_t) \quad \text{for} \quad i\Delta_t \le \zeta \le (i+1)\Delta_t. \tag{10.88}$$

Using the pulse function, the field over all values of ζ can be written

$$\mathbf{E}(q\Delta_t - \zeta) = \sum_{i=0}^{q-1} \left[\mathbf{E}^{q-i} + \frac{\mathbf{E}^{q-i-1} - \mathbf{E}^{q-i}}{\Delta_t}(\zeta - i\Delta_t) \right] p_i(\zeta). \tag{10.89}$$

Note the limits of the summation. The upper limit of integration is $q\Delta_t$ which corresponds to the end-point of the segment which varies from $(q-1)\Delta_t$ to $q\Delta_t$. This segment has an index of $q-1$ (segment 0 varies from 0 to Δ_t, segment 1 varies from Δ_t to $2\Delta_t$, and so on). The lower limit used here is not actually dictated by the electric field. Rather, when we combine the electric field with the susceptibility function the product is zero for ζ less than zero since the susceptibility is zero for ζ less than zero (due to the material impulse response being causal). Hence we start the lower limit of the summation at zero.

Substituting (10.89) into (10.84) yields

$$\mathbf{D}^q = \epsilon_0\epsilon_\infty \mathbf{E}^q + \epsilon_0 \int\limits_{\zeta=0}^{q\Delta_t} \sum_{i=0}^{q-1} \left[\mathbf{E}^{q-i} + \frac{\mathbf{E}^{q-i-1} - \mathbf{E}^{q-i}}{\Delta_t}(\zeta - i\Delta_t) \right] p_i(\zeta)\chi_e(\zeta)d\zeta. \tag{10.90}$$

The summation and integration can be interchanged. However, the pulse function dictates that the integration only needs to be carried out over the range of values where the pulse function is unity. Thus we can write

$$\mathbf{D}^q = \epsilon_0\epsilon_\infty \mathbf{E}^q + \epsilon_0 \sum_{i=0}^{q-1} \int\limits_{\zeta=i\Delta_t}^{(i+1)\Delta_t} \left[\mathbf{E}^{q-i} + \frac{\mathbf{E}^{q-i-1} - \mathbf{E}^{q-i}}{\Delta_t}(\zeta - i\Delta_t) \right] \chi_e(\zeta)d\zeta. \tag{10.91}$$

The samples of the electric field are constants with respect to the variable of integration and can be taken outside of the integral. This yields

$$\begin{aligned}\mathbf{D}^q &= \epsilon_0\epsilon_\infty \mathbf{E}^q + \\ &\epsilon_0 \sum_{i=0}^{q-1} \left[\mathbf{E}^{q-i} \left(\int\limits_{\zeta=i\Delta_t}^{(i+1)\Delta_t} \chi_e(\zeta)d\zeta \right) + \frac{\mathbf{E}^{q-i-1} - \mathbf{E}^{q-i}}{\Delta_t} \left(\int\limits_{\zeta=i\Delta_t}^{(i+1)\Delta_t} (\zeta - i\Delta_t)\chi_e(\zeta)d\zeta \right) \right]\end{aligned} \tag{10.92}$$

To simplify the notation, we define the following

$$\chi^i = \int_{\zeta=i\Delta_t}^{(i+1)\Delta_t} \chi_e(\zeta)d\zeta, \tag{10.93}$$

$$\xi^i = \frac{1}{\Delta_t}\int_{\zeta=i\Delta_t}^{(i+1)\Delta_t} (\zeta - i\Delta_t)\chi_e(\zeta)d\zeta. \tag{10.94}$$

This allows us to write

$$\mathbf{D}^q = \epsilon_0\epsilon_\infty\mathbf{E}^q + \epsilon_0\sum_{i=0}^{q-1}\left[\mathbf{E}^{q-i}\chi^i + \left(\mathbf{E}^{q-i-1} - \mathbf{E}^{q-i}\right)\xi^i\right]. \tag{10.95}$$

In discrete form and using the electric flux density, Ampere's law can be written

$$\nabla\times\mathbf{H}^{q+1/2} = \frac{\mathbf{D}^{q+1} - \mathbf{D}^q}{\Delta_t}. \tag{10.96}$$

Equation (10.95) gives $\mathbf{D}^q$ in terms of the electric field. This equation can also be used to express $\mathbf{D}^{q+1}$ in terms of the electric field: one merely replaces q with $q+1$. This yields

$$\mathbf{D}^{q+1} = \epsilon_0\epsilon_\infty\mathbf{E}^{q+1} + \epsilon_0\sum_{i=0}^{q}\left[\mathbf{E}^{q-i+1}\chi^i + \left(\mathbf{E}^{q-i} - \mathbf{E}^{q-i+1}\right)\xi^i\right]. \tag{10.97}$$

In order to obtain an update equation for the electric field, we must express $\mathbf{E}^{q+1}$ in terms of other known (or past) quantities. As things stand now, there is an $\mathbf{E}^{q+1}$ "buried" inside the the summation in (10.97). To express that explicitly, we extract the $i = 0$ term and then have the summation start from $i = 1$. This yields

$$\begin{aligned}\mathbf{D}^{q+1} &= \epsilon_0\epsilon_\infty\mathbf{E}^{q+1} + \epsilon_0\mathbf{E}^{q+1}\chi^0 + \epsilon_0\left(\mathbf{E}^q - \mathbf{E}^{q+1}\right)\xi^0 \\ &\quad +\epsilon_0\sum_{i=1}^{q}\left[\mathbf{E}^{q-i+1}\chi^i + \left(\mathbf{E}^{q-i} - \mathbf{E}^{q-i+1}\right)\xi^i\right]. \end{aligned} \tag{10.98}$$

Ultimately we want to combine the summations in (10.95) and (10.98) and thus the limits of the summations must be the same. The limits of the summation in (10.98) can be adjusting by using a new index $i' = i - 1$ (thus $i = i' + 1$). Substituting i' for i, (10.98) can be written

$$\begin{aligned}\mathbf{D}^{q+1} &= \mathbf{E}^{q+1}\epsilon_0(\epsilon_\infty + \chi^0 - \xi^0) + \mathbf{E}^q\epsilon_0\xi^0 \\ &\quad +\epsilon_0\sum_{i'=0}^{q-1}\left[\mathbf{E}^{q-i'}\chi^{i'+1} + \left(\mathbf{E}^{q-i'-1} - \mathbf{E}^{q-i'}\right)\xi^{i'+1}\right]. \end{aligned} \tag{10.99}$$

Since i' is just an index, we can return to calling is merely i.

The temporal finite-difference of the flux density is obtained by combining (10.95) and (10.99). The result is

$$\begin{aligned}\frac{\mathbf{D}^{q+1}-\mathbf{D}^{q}}{\Delta_t} &= \frac{1}{\Delta_t}\Bigg(\mathbf{E}^{q+1}\epsilon_0(\epsilon_\infty-\chi^0+\xi^0)+\mathbf{E}^{q}\epsilon_0(-\epsilon_\infty+\xi^0)\\ &\qquad -\epsilon_0\sum_{i=0}^{q-1}\left[\mathbf{E}^{q-i}\Delta\chi^i+\left(\mathbf{E}^{q-i-1}-\mathbf{E}^{q-i}\right)\Delta\xi^i\right]\Bigg) \qquad (10.100)\end{aligned}$$

where

$$\begin{aligned}\Delta\chi^i &= \chi^i-\chi^{i+1}, &(10.101)\\ \Delta\xi^i &= \xi^i-\xi^{i+1}. &(10.102)\end{aligned}$$

The summation in (10.100) does not contain $\mathbf{E}^{q+1}$. Hence, using (10.100) to replace the right side of (10.96) and solving for $\mathbf{E}^{q+1}$ yields

$$\mathbf{E}^{q+1}=\frac{\epsilon_\infty-\xi^0}{\epsilon_\infty+\chi^0-\xi^0}\mathbf{E}^q+\frac{\frac{\Delta_t}{\epsilon_0}}{\epsilon_\infty+\chi^0-\xi^0}\nabla\times\mathbf{H}^{q+1/2}+\frac{1}{\epsilon_\infty+\chi^0-\xi^0}\mathbf{\Psi}^q \qquad (10.103)$$

where $\mathbf{\Psi}^q$, known as the recursive accumulator, is given by

$$\mathbf{\Psi}^q=\sum_{i=0}^{q-1}\left[\mathbf{E}^{q-i}\Delta\chi^i+\left(\mathbf{E}^{q-i-1}-\mathbf{E}^{q-i}\right)\Delta\xi^i\right] \qquad (10.104)$$

Equation (10.103) is used to update the electric field. It appears that a summation must be computed which requires knowledge of all the previous values of the electric field. Clearly this would be prohibitive if this were the case in practice. Fortunately, provided the material impulse response can be expressed in terms of exponentials, there is a recursive formulation which can be used to efficiently express this summation.

Consider $\mathbf{\Psi}^q$ with the $i=0$ term written explicitly, i.e.,

$$\mathbf{\Psi}^q=\mathbf{E}^q(\Delta\chi^0-\Delta\xi^0)+\mathbf{E}^{q-1}\Delta\xi^0+\sum_{i=1}^{q-1}\left[\mathbf{E}^{q-i}\Delta\chi^i+\left(\mathbf{E}^{q-i-1}-\mathbf{E}^{q-i}\right)\Delta\xi^i\right]. \qquad (10.105)$$

Employing a change of indices for the summation so that the new limits range from 0 to $q-2$, this can be written:

$$\mathbf{\Psi}^q=\mathbf{E}^q(\Delta\chi^0-\Delta\xi^0)+\mathbf{E}^{q-1}\Delta\xi^0+\sum_{i=0}^{q-2}\left[\mathbf{E}^{q-i-1}\Delta\chi^{i+1}+\left(\mathbf{E}^{q-i-2}-\mathbf{E}^{q-i-1}\right)\Delta\xi^{i+1}\right]. \qquad (10.106)$$

Now consider $\mathbf{\Psi}^{q-1}$ by writing (10.104) with q replaced by $q-1$:

$$\mathbf{\Psi}^{q-1}=\sum_{i=0}^{q-2}\left[\mathbf{E}^{q-i-1}\Delta\chi^i+\left(\mathbf{E}^{q-i-2}-\mathbf{E}^{q-i-1}\right)\Delta\xi^i\right] \qquad (10.107)$$

Note the similarity between the summation in (10.106) and the right-hand side of (10.107). These are the same except in (10.106) the summation involves $\Delta\chi^{i+1}$ and $\Delta\xi^{i+1}$ while in (10.107) the summation involves $\Delta\chi^i$ and $\Delta\xi^i$. As we will see, for certain materials it is possible to relate these values to each other in a simple way. Specifically, we will find that these are related by

$$\begin{aligned}\Delta\chi^{i+1} &= C_{rec}\Delta\chi^i, &(10.108)\\ \Delta\xi^{i+1} &= C_{rec}\Delta\xi^i, &(10.109)\end{aligned}$$

where C_{rec} is a "recursion constant" (which is yet to be determined). Given that this recursion relationship exists for $\Delta\chi^{i+1}$ and $\Delta\xi^{i+1}$, (10.106) can be written

$$\begin{aligned}\mathbf{\Psi}^q &= \mathbf{E}^q(\Delta\chi^0 - \Delta\xi^0) + \mathbf{E}^{q-1}\Delta\xi^0 + C_{rec}\sum_{i=0}^{q-2}\left[\mathbf{E}^{q-i-1}\Delta\chi^i + \left(\mathbf{E}^{q-i-2} - \mathbf{E}^{q-i-1}\right)\Delta\xi^i\right],\\ &= \mathbf{E}^q(\Delta\chi^0 - \Delta\xi^0) + \mathbf{E}^{q-1}\Delta\xi^0 + C_{rec}\mathbf{\Psi}^{q-1}, \qquad (10.110)\end{aligned}$$

or, after replacing q with $q+1$, this becomes

$$\mathbf{\Psi}^{q+1} = \mathbf{E}^{q+1}(\Delta\chi^0 - \Delta\xi^0) + \mathbf{E}^q\Delta\xi^0 + C_{rec}\mathbf{\Psi}^q. \qquad (10.111)$$

The PLRC algorithm is now, at least in the abstract sense, complete. The implementation is as follows:

1. Update the magnetic field in the usual way (perhaps using a dispersive formulation).
2. Update the electric field using (10.103) (being sure to first store the previous value of the electric field).
3. Updated the recursive accumulator as specified by (10.111) (using both the updated electric field and the stored value).
4. Repeat.

It now remains to determine the various constants for a given material. Specifically, one must know χ^0, ξ^0, ϵ_∞ (which appear in (10.103)), as well as $\Delta\chi^0$, $\Delta\xi^0$, and C_{rec} (which appear in (10.111)).

10.7 PLRC for Debye Material

The time-domain form of the susceptibility function for Deybe materials was given in (10.29). Using this in (10.93) and (10.94) yields

$$\begin{aligned}\chi^i &= \epsilon_d\left(1 - e^{-\Delta_t/\tau_d}\right)e^{-i\Delta_t/\tau_d}, &(10.112)\\ \xi^i &= \frac{\epsilon_d\tau_d}{\Delta_t}\left(1 - \left[\frac{\Delta_t}{\tau_d} + 1\right]e^{-\Delta_t/\tau_d}\right)e^{-i\Delta_t/\tau_d}. &(10.113)\end{aligned}$$

Setting i equal to zero yields

$$\begin{aligned}\chi^0 &= \epsilon_d\left(1 - e^{-\Delta_t/\tau_d}\right), &(10.114)\\ \xi^0 &= \frac{\epsilon_d\tau_d}{\Delta_t}\left(1 - \left[\frac{\Delta_t}{\tau_d} + 1\right]e^{-\Delta_t/\tau_d}\right). &(10.115)\end{aligned}$$

The time-constant τ_d can be expressed in terms of multiples of the time step Δ_t, e.g., $\tau_d = N_d\Delta_t$. Note that the time step is always divided by τ_d in these expressions so that the only important consideration is the ratio of these quantities, i.e., $\Delta_t/\tau_d = 1/N_d$.

From (10.112) we observe

$$\begin{aligned}
\chi^{i+1} &= \epsilon_d \left(1 - e^{-\Delta_t/\tau_d}\right) e^{-(i+1)\Delta_t/\tau_d}, \\
&= e^{-\Delta_t/\tau_d}\epsilon_d \left(1 - e^{-\Delta_t/\tau_d}\right) e^{-i\Delta_t/\tau_d}, \\
&= e^{-\Delta_t/\tau_d}\chi^i.
\end{aligned} \tag{10.116}$$

Now consider $\Delta\chi^i$ which is given by

$$\begin{aligned}
\Delta\chi^i &= \chi^i - \chi^{i+1}, \\
&= \chi^i - e^{-\Delta_t/\tau_d}\chi^i, \\
&= \chi^i(1 - e^{-\Delta_t/\tau_d}).
\end{aligned} \tag{10.117}$$

Thus $\Delta\chi^{i+1}$ can be written as

$$\begin{aligned}
\Delta\chi^{i+1} &= \chi^{i+1}(1 - e^{-\Delta_t/\tau_d}), \\
&= e^{-\Delta_t/\tau_d}\chi^i(1 - e^{-\Delta_t/\tau_d}), \\
&= e^{-\Delta_t/\tau_d}\Delta\chi^i.
\end{aligned} \tag{10.118}$$

Similar arguments pertain to ξ^i and $\Delta\xi^i$ resulting in

$$\Delta\xi^{i+1} = e^{-\Delta_t/\tau_d}\Delta\xi^i. \tag{10.119}$$

From (10.118) and (10.119), and in accordance with the description of C_{rec} given in (10.108) and (10.109), we conclude that

$$C_{rec} = e^{-\Delta_t/\tau_d}. \tag{10.120}$$

The factor $\exp(-\Delta_t/\tau_d)$, i.e., C_{rec}, appears in several of the terms given above. Writing the ratio Δ_t/τ_d as $1/N_d$, all the terms involved in the implementation of the PLRC method can be expressed as

$$C_{rec} = e^{-1/N_d}, \tag{10.121}$$

$$\chi^0 = \epsilon_d \left(1 - C_{rec}\right), \tag{10.122}$$

$$\xi^0 = \epsilon_d N_d \left(1 - \left[\frac{1}{N_d} + 1\right] C_{rec}\right), \tag{10.123}$$

$$\Delta\chi^0 = \chi^0(1 - C_{rec}), \tag{10.124}$$

$$\Delta\xi^0 = \xi^0(1 - C_{rec}). \tag{10.125}$$

Nearly all the terms involved in the PLRC method are unitless and independent of scale. The only term which is not is Δ_t/ϵ_0 which multiplies the curl of the magnetic field in (10.103). However, this is a term which has appeared in all the electric-field update equations we have ever considered and, after extracting the $1/\delta$ inherent in the finite-difference form of the curl operator, it can be expressed in terms of the Courant number and characteristic impedance, i.e., $\Delta_t/(\delta\epsilon_0) = S_c\eta_0$.