

# Chapter 12

## Acoustic FDTD Simulations

### 12.1 Introduction

The FDTD method employs finite-differences to approximate Ampere’s and Faraday’s laws. Ampere’s and Faraday’s laws are first-order differential equations that couple the electric and magnetic fields. As we have seen, with a judicious discretization of space and time, the resulting equations can be solved for “future” fields in terms of known past fields.

Other physical phenomena are also described by coupled first-order differential equations where the temporal derivative of one field is related to the spatial derivative of another field. Both acoustics and elastic wave propagation are such phenomena. Here we will consider only acoustic propagation. Specifically we will consider small-signal acoustics which can be described in terms of the scalar pressure field  $P(x, y, z, t)$  and the vector velocity  $\mathbf{v}(x, y, z, t)$ . The material parameters are the speed of sound  $c_a$  and the density  $\rho$  (both of which can vary as a function of position).

The governing acoustic equations in three dimensions are

$$\frac{\partial P}{\partial t} = -\rho c_a^2 \nabla \cdot \mathbf{v}, \quad (12.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla P, \quad (12.2)$$

or, expanded in terms of the components,

$$\frac{\partial P}{\partial t} = -\rho c_a^2 \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right), \quad (12.3)$$

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial x}, \quad (12.4)$$

$$\frac{\partial v_y}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial y}, \quad (12.5)$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z}. \quad (12.6)$$

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<sup>†</sup>Lecture notes by John Schneider. ftdt-acoustics.tex

Equation (12.2) is essentially a variation of Newton's second law,  $\mathbf{F} = m\mathbf{a}$ , where instead of acceleration  $\mathbf{a}$  there is the derivative of the velocity, instead of mass  $m$  there is the mass density, and instead of force  $\mathbf{F}$  there is the derivative of pressure. Pressure is force per area and the negative sign accounts for the fact that if pressure is building in a particular direction that tends to cause acceleration in the opposite direction. Equation (12.1) comes from an equation of state for the material (with various approximations assumed along the way).

Taking the divergence of (12.2) and interchanging the order of temporal and spatial differentiation yields

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{v} = -\frac{1}{\rho} \nabla^2 P. \quad (12.7)$$

Taking the temporal derivative of (12.1) and using (12.7) yields

$$\frac{\partial^2 P}{\partial t^2} = -\rho c_a^2 \frac{\partial}{\partial t} \nabla \cdot \mathbf{v} = c_a^2 \nabla^2 P. \quad (12.8)$$

Rearranging this yields the wave equation

$$\nabla^2 P - \frac{1}{c_a^2} \frac{\partial^2 P}{\partial t^2} = 0. \quad (12.9)$$

Thus the usual techniques and solutions one is familiar with from electromagnetics carry over to acoustics. For example, a harmonic plane wave given by

$$P(x, y, z, t) = P_0 e^{-j\boldsymbol{\beta} \cdot \mathbf{r}} e^{j\omega t} \quad (12.10)$$

is a valid solution to the governing equations where  $P_0$  is a constant and the wave vector  $\boldsymbol{\beta}$  can be written

$$\boldsymbol{\beta} = \beta_x \hat{\mathbf{a}}_x + \beta_y \hat{\mathbf{a}}_y + \beta_z \hat{\mathbf{a}}_z = \beta \hat{\mathbf{a}}_\beta = (\omega/c_a) \hat{\mathbf{a}}_\beta. \quad (12.11)$$

Substituting (12.10) into (12.4) and assuming  $\exp(j\omega t)$  temporal dependence yields

$$j\omega v_x = \frac{1}{\rho} (-j\beta_x) P. \quad (12.12)$$

Rearranging terms gives

$$v_x = \frac{\beta_x}{\rho\omega} P. \quad (12.13)$$

Following the same steps for the  $y$  and  $z$  components produces

$$v_y = \frac{\beta_y}{\rho\omega} P, \quad (12.14)$$

$$v_z = \frac{\beta_z}{\rho\omega} P. \quad (12.15)$$

Thus the harmonic velocity is given by

$$\mathbf{v} = v_x \hat{\mathbf{a}}_x + v_y \hat{\mathbf{a}}_y + v_z \hat{\mathbf{a}}_z = \frac{1}{\rho\omega} (\beta_x \hat{\mathbf{a}}_x + \beta_y \hat{\mathbf{a}}_y + \beta_z \hat{\mathbf{a}}_z) P = \frac{\beta}{\rho\omega} P \hat{\mathbf{a}}_\beta. \quad (12.16)$$

Since the wave number, i.e., the magnitude of the wave vector, is given by  $\beta = \omega/c_a$ , the ratio of the magnitude of pressure to the velocity is given by

$$\left| \frac{P}{\mathbf{v}} \right| = \rho c_a. \quad (12.17)$$

The term on the right-hand side is known as the characteristic impedance of the medium which is often written as  $Z$ .

## 12.2 Governing FDTD Equations

To obtain an FDTD algorithm for acoustic propagation, the pressure and components of velocity are discretized in both time and space. In electromagnetics there were two vector fields and hence six field-components that had to be arranged in space-time. In acoustics there is one scalar field and one vector field. Thus there are only four field-components.

To implement a 3D acoustic FDTD algorithm, a suitable arrangement of nodes is as shown in Fig. 12.1. A pressure node is surrounded by velocity components such that the components are oriented along the line joining the component and the pressure node. This should be contrasted to the arrangement of nodes in electromagnetic grids where the components of the magnetic field swirled around the components of the electric field, and vice versa. In electromagnetics one is modeling coupled curl equations where the partial derivatives are related to behavior orthogonal to the direction of the derivative. In acoustics, where the governing equations involve the divergence and gradient, the partial derivatives are associated with behavior in the direction of the derivative.

The arrangement of nodes in a 2D grid is illustrated in Fig. 12.2. This should be compared to the 2D electromagnetic grids, i.e., Fig. 8.1 for the  $TM^z$  case and Fig. 8.9 for the  $TE^z$  case. (Because pressure is inherently a scalar field, there are not two different polarization associated with 2D acoustic simulations—nor is there a notion of polarization in three dimensions.)

In addition to the spatial offsets, the pressure nodes are assumed to be offset a half temporal step from the velocity nodes (but all the velocity components exist at the same time-step). The following notation will be used with an implicit understanding of spatial offsets

$$P(x, y, z, t) = P(m\Delta_x, n\Delta_y, p\Delta_z, q\Delta_t) = P^q[m, n, p], \quad (12.18)$$

$$v_x(x, y, z, t) = v_x([m + 1/2]\Delta_x, n\Delta_y, p\Delta_z, [q + 1/2]\Delta_t) = v_x^{q+1/2}[m, n, p], \quad (12.19)$$

$$v_y(x, y, z, t) = v_y(m\Delta_x, [n + 1/2]\Delta_y, p\Delta_z, [q + 1/2]\Delta_t) = v_y^{q+1/2}[m, n, p], \quad (12.20)$$

$$v_z(x, y, z, t) = v_z(m\Delta_x, n\Delta_y, [p + 1/2]\Delta_z, [q + 1/2]\Delta_t) = v_z^{q+1/2}[m, n, p]. \quad (12.21)$$

We will assume the spatial step sizes are the same, i.e.,  $\Delta_x = \Delta_y = \Delta_z = \delta$ .

Replacing the derivatives in (12.3) with finite differences and using the discretization of (12.18)–(12.21) yields the following update equation:

$$\begin{aligned} P^q[m, n, p] = & P^{q-1}[m, n, p] - \rho c_a^2 \frac{\Delta_t}{\delta} \left( v_x^{q-1/2}[m, n, p] - v_x^{q-1/2}[m-1, n, p] + \right. \\ & v_y^{q-1/2}[m, n, p] - v_y^{q-1/2}[m, n-1, p] + \\ & \left. v_z^{q-1/2}[m, n, p] - v_z^{q-1/2}[m, n, p-1] \right). \quad (12.22) \end{aligned}$$

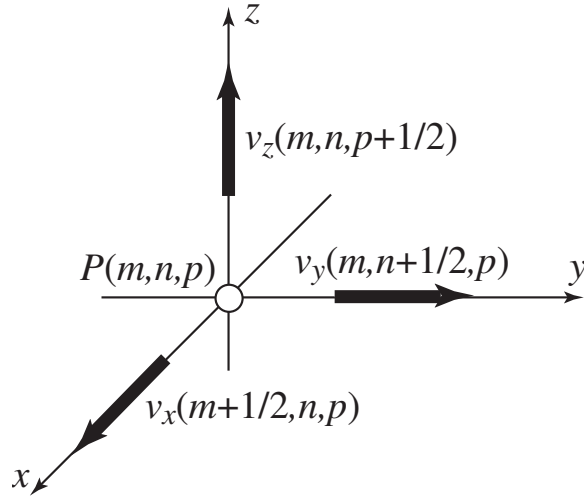


Figure 12.1: An acoustic unit cell in three dimensions showing the arrangement of velocity nodes relative to the pressure node with the same spatial indices.

The sound speed and the density can be functions of space. Let us assume that the density and sound speed are specified at the grid points corresponding to the location of pressure nodes. Additionally, assume that the sound speed can be defined in terms of a background sound speed  $c_0$  and a relative sound speed  $c_r$ :

$$c_a = c_r c_0. \quad (12.23)$$

The background sound speed corresponds to the fastest speed of propagation at any location in the grid so that  $c_r \leq 1$ . The coefficient of the spatial finite-difference in (12.22) can now be written

$$\rho c_a^2 \frac{\Delta t}{\delta} = \rho c_r^2 c_0 \frac{c_0 \Delta t}{\delta} = \rho c_r^2 c_0 S_c \quad (12.24)$$

where, similar to electromagnetics, the Courant number is  $S_c = c_0 \Delta t / \delta$ . The explicit spatial dependence of the density and sound speed can be emphasized by writing the coefficient as

$$\rho[m, n, p] c_r^2[m, n, p] c_0 S_c \quad (12.25)$$

where  $\rho[m, n, p]$  is the density that exists at the same point as the pressure node  $P[m, n, p]$  and  $c_r[m, n, p]$  is the relative sound speed at this same point. Note that the Courant number  $S_c$  and the background sound speed  $c_0$  are independent of position. Furthermore, the entire coefficient is independent of time.

The update equation for the  $x$  component of velocity is obtained from the discretized version of (12.4) which yields

$$v_x^{q+1/2}[m, n, p] = v_x^{q-1/2}[m, n, p] - \frac{1}{\rho} \frac{\Delta t}{\delta} (P^q[m+1, n, p] - P^q[m, n, p]) \quad (12.26)$$

The coefficient of this equation does not contain the Courant number but that can be obtained by multiplying and dividing by the background sound speed

$$\frac{1}{\rho} \frac{\Delta t}{\delta} = \frac{1}{\rho c_0} \frac{c_0 \Delta t}{\delta} = \frac{1}{\rho c_0} S_c. \quad (12.27)$$

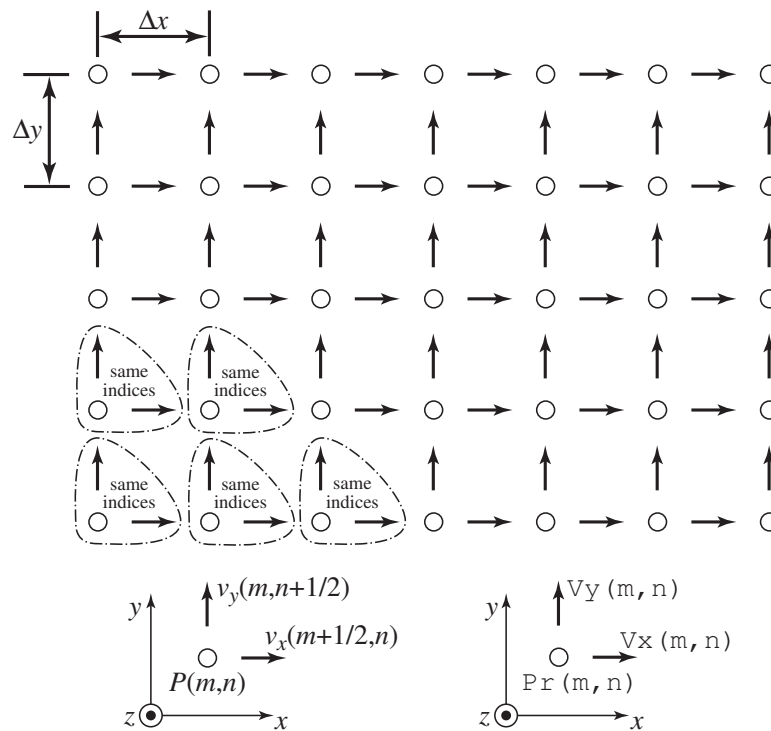


Figure 12.2: The arrangement of nodes in a 2D acoustic simulation. In a computer FDTD implementation the nodes shown within the dashed enclosures will have the same spatial indices. This is illustrated by the two depictions of a unit cell at the bottom of the figure. The one on the left shows the nodes with the spatial offsets given explicitly. The one on the right shows the corresponding node designations that would be used in a computer program. (Here  $P_r$  is used for the pressure array.)

We wish to define the density only at the pressure nodes. Since the  $x$ -component of the velocity is offset from the pressure a half spatial step in the  $x$  direction, what is the appropriate velocity to use? The answer, much as it was in the case of an interface between two different materials in electromagnetics, is the average of the densities to either side of the pressure node (where the notion of “either side” is dictated by the orientation of the velocity node). Therefore the coefficient can be written

$$\frac{1}{\left(\frac{\rho[m+1,n,p]+\rho[m,n,p]}{2}\right)c_0}S_c = \frac{2S_c}{(\rho[m+1,n,p] + \rho[m,n,p])c_0}. \quad (12.28)$$

The update equations for the velocity components can now be written as

$$v_x^{q+1/2}[m,n,p] = v_x^{q-1/2}[m,n,p] - \frac{2S_c}{(\rho[m,n,p] + \rho[m+1,n,p])c_0} (P^q[m+1,n,p] - P^q[m,n,p]), \quad (12.29)$$

$$v_y^{q+1/2}[m,n,p] = v_y^{q-1/2}[m,n,p] - \frac{2S_c}{(\rho[m,n,p] + \rho[m,n+1,p])c_0} (P^q[m,n+1,p] - P^q[m,n,p]), \quad (12.30)$$

$$v_z^{q+1/2}[m,n,p] = v_z^{q-1/2}[m,n,p] - \frac{2S_c}{(\rho[m,n,p] + \rho[m,n,p+1])c_0} (P^q[m,n,p+1] - P^q[m,n,p]). \quad (12.31)$$

## 12.3 Two-Dimensional Implementation

Let us consider a 2D simulation in which the fields vary in the  $x$  and  $y$  directions. The grid would be as shown in Fig. 12.2 and it is assumed that  $\Delta_x = \Delta_y = \delta$ . Assume the arrays  $\text{pr}$ ,  $\text{vx}$ , and  $\text{vy}$  hold the pressure,  $x$  component of the velocity, and the  $y$  component of the velocity, respectively. Similar to the electromagnetic implementation, assume the macros  $\text{Pr}$ ,  $\text{Vx}$ , and  $\text{Vy}$  have been created to facilitate accessing these arrays (ref. Sec. 8.2). The update equations can be written

$$\begin{aligned} \text{Vx}(m,n) &= \text{Vx}(m,n) - C_{vxp}(m,n) * (\text{Pr}(m+1,n) - \text{Pr}(m,n)); \\ \text{Vy}(m,n) &= \text{Vy}(m,n) - C_{vyp}(m,n) * (\text{Pr}(m,n+1) - \text{Pr}(m,n)); \\ \text{Pr}(m,n) &= \text{Pr}(m,n) - C_{prv}(m,n) * ((\text{Vx}(m,n) - \text{Vx}(m-1,n)) \\ &\quad + (\text{Vy}(m,n) - \text{Vy}(m,n-1))); \end{aligned}$$

where the coefficient arrays are given by

$$C_{vxp}(m,n) = \frac{1}{\rho c_0} S_c \Big|_{(m+1/2)\delta, n\delta} = \frac{2S_c}{(\rho[m+1,n] + \rho[m,n])c_0}, \quad (12.32)$$

$$C_{vyp}(m,n) = \frac{1}{\rho c_0} S_c \Big|_{m\delta, (n+1/2)\delta} = \frac{2S_c}{(\rho[m,n] + \rho[m,n+1])c_0}, \quad (12.33)$$

$$C_{prv}(m,n) = \rho[m,n]c_r^2[m,n]c_0S_c. \quad (12.34)$$

These update equations are little different from those for the  $\text{TM}^z$  case. Referring to Sec. 8.3 for lossless materials, the  $\text{TM}^z$  update equations are

$$\begin{aligned}
H_y(m, n) &= H_y(m, n) + C_{hye}(m, n) * (E_z(m+1, n) - E_z(m, n)); \\
H_x(m, n) &= H_x(m, n) - C_{hxe}(m, n) * (E_z(m, n+1) - E_z(m, n)); \\
E_z(m, n) &= E_z(m, n) + C_{ezh}(m, n) * ((H_y(m, n) - H_y(m-1, n)) \\
&\quad - (H_x(m, n) - H_x(m, n-1)));
\end{aligned}$$

There is a one-to-one mapping between these sets of equations. One can equate values as follows

$$v_x \Leftrightarrow -H_y, \quad (12.35)$$

$$v_y \Leftrightarrow H_x, \quad (12.36)$$

$$P \Leftrightarrow E_z, \quad (12.37)$$

$$C_{vxp} \Leftrightarrow C_{hye}, \quad (12.38)$$

$$C_{vyp} \Leftrightarrow C_{hxe}, \quad (12.39)$$

$$C_{prv} \Leftrightarrow C_{ezh}. \quad (12.40)$$

Thus, converting 2D programs that were written to model electromagnetic field propagation to ones that can model acoustic propagation is surprisingly straightforward. Essentially, all one has to do is change some labels and a few signs.

For  $TE^z$  simulations, the updated equations for a lossless medium were

$$\begin{aligned}
H_z(m, n) &= H_z(m, n) + \\
&\quad C_{hze}(m, n) * ((E_x(m, n+1) - E_x(m, n)) - (E_y(m+1, n) - E_y(m, n))); \\
E_x(m, n) &= E_x(m, n) + C_{exh}(m, n) * (H_z(m, n) - H_z(m, n-1)); \\
E_y(m, n) &= E_y(m, n) - C_{eyh}(m, n) * (H_z(m, n) - H_z(m-1, n));
\end{aligned}$$

In this case the conversion from the electromagnetic equations to the acoustic equations can be accomplished with the following mapping

$$v_x \Leftrightarrow E_y, \quad (12.41)$$

$$v_y \Leftrightarrow -E_x, \quad (12.42)$$

$$P \Leftrightarrow H_z, \quad (12.43)$$

$$C_{vxp} \Leftrightarrow C_{eyh}, \quad (12.44)$$

$$C_{vyp} \Leftrightarrow C_{exh}, \quad (12.45)$$

$$C_{prv} \Leftrightarrow C_{hze}. \quad (12.46)$$

For three dimensions 3D acoustic code is arguably simpler than the electromagnetic case since there are not two vector fields. However porting 3D electromagnetic algorithms to the acoustic case is not as trivial as in two dimensions.

