EE331—EXAMPLE #13: COORDINATE TRANSFORMATION

Transform the vector field $\mathbf{A} = \frac{y^2}{x^2+y^2}\widehat{\mathbf{a}}_x - \frac{x^2}{x^2+y^2}\widehat{\mathbf{a}}_y + 4\widehat{\mathbf{a}}_z$ into cylindrical coordinates.

We're given a vector field of the form:

$$\mathbf{A} = A_x\widehat{\mathbf{a}}_x + A_y\widehat{\mathbf{a}}_y + A_z\widehat{\mathbf{a}}_z$$

that we want to transform into a vector field of the form:

$$\mathbf{A} = A_\rho\widehat{\mathbf{a}}_\rho + A_\phi\widehat{\mathbf{a}}_\phi + A_z\widehat{\mathbf{a}}_z$$

For the vector projection approach, there are two steps. First we transform the components A_x, A_y, and A_z; then we transform the variables x, y, and z.

1. Transform components:

$$\begin{aligned} A_\rho = \mathbf{A}\cdot\widehat{\mathbf{a}}_\rho &= \frac{y^2}{x^2+y^2}\widehat{\mathbf{a}}_x\cdot\widehat{\mathbf{a}}_\rho - \frac{x^2}{x^2+y^2}\widehat{\mathbf{a}}_y\cdot\widehat{\mathbf{a}}_\rho + 4\widehat{\mathbf{a}}_z\cdot\widehat{\mathbf{a}}_\rho \\ &= \frac{y^2}{x^2+y^2}\cos\phi - \frac{x^2}{x^2+y^2}\sin\phi + 0 \end{aligned}$$

   where we've used the tables at the bottom of the yellow sheet for the dot products of the unit vectors.

$$\begin{aligned} A_\phi = \mathbf{A}\cdot\widehat{\mathbf{a}}_\phi &= \frac{y^2}{x^2+y^2}\widehat{\mathbf{a}}_x\cdot\widehat{\mathbf{a}}_\phi - \frac{x^2}{x^2+y^2}\widehat{\mathbf{a}}_y\cdot\widehat{\mathbf{a}}_\phi + 4\widehat{\mathbf{a}}_z\cdot\widehat{\mathbf{a}}_\phi \\ &= -\frac{y^2}{x^2+y^2}\sin\phi - \frac{x^2}{x^2+y^2}\cos\phi + 0 \end{aligned}$$

$$\begin{aligned} A_z = \mathbf{A}\cdot\widehat{\mathbf{a}}_z &= \frac{y^2}{x^2+y^2}\widehat{\mathbf{a}}_x\cdot\widehat{\mathbf{a}}_z - \frac{x^2}{x^2+y^2}\widehat{\mathbf{a}}_y\cdot\widehat{\mathbf{a}}_z + 4\widehat{\mathbf{a}}_z\cdot\widehat{\mathbf{a}}_z \\ &= 0 - 0 + 4 \end{aligned}$$

2. Transform variables:

$$\frac{y^2}{x^2+y^2} = \frac{\rho^2\sin^2\phi}{\rho^2} = \sin^2\phi$$
$$\frac{x^2}{x^2+y^2} = \frac{\rho^2\cos^2\phi}{\rho^2} = \cos^2\phi$$

   where we've used the conversions at the bottom of the yellow sheet. Thus,

$$A_\rho = \sin^2\phi\cos\phi - \cos^2\phi\sin\phi = \sin\phi\cos\phi(\sin\phi - \cos\phi)$$
$$A_\phi = -\sin^3\phi - \cos^3\phi = -(\sin^3\phi + \cos^3\phi)$$
$$A_z = 4$$

   Finally, then,

$$\mathbf{A} = \sin\phi\cos\phi(\sin\phi - \cos\phi)\widehat{\mathbf{a}}_\rho - (\sin^3\phi + \cos^3\phi)\widehat{\mathbf{a}}_\phi + 4\widehat{\mathbf{a}}_z$$

Note that in the final result there are no rectangular variables (and there shouldn't be!).