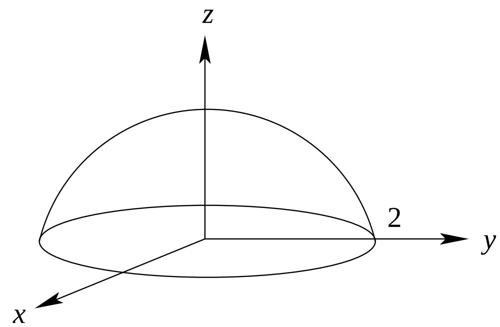
Given the vector field $\mathbf{B}(r,\theta,\phi) = r^2\cos\phi\,\widehat{\mathbf{a}}_r + r\cos\phi\,\widehat{\mathbf{a}}_\theta$, find the net outward flux through the closed hemisphere $0 \le r \le 2$, $0 \le \theta \le \pi/2$, and $0 \le \phi < 2\pi$.

First make a sketch of the surface. (Sorry about the strange perspective. It was the best I could do.)



There are two surfaces over which we must integrate, the top of the hemisphere and the bottom of the hemisphere.

$$\oint \mathbf{B}\cdot\mathbf{dS} = \int_{top} \mathbf{B}\cdot\mathbf{dS} + \int_{bottom} \mathbf{B}\cdot\mathbf{dS}$$

First integrate over the top:

$$\begin{aligned}
\int_{top} \mathbf{B}\cdot\mathbf{dS} &= \int \mathbf{B}\cdot(r^2\sin\theta d\theta d\phi\,\widehat{\mathbf{a}}_r) \\
&= \int B_r(r^2\sin\theta d\theta d\phi) = \int (r^2\cos\phi)(r^2\sin\theta d\theta d\phi) \\
&= r^4\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{2\pi}\cos\phi\,d\phi = (2)^4\cos\theta\,|_{\pi/2}^{0}\,\sin\phi\,|_0^{2\pi} \\
&= 0
\end{aligned}$$

Next integrate over the bottom:

$$\begin{aligned}
\int_{bottom} \mathbf{B}\cdot\mathbf{dS} &= \int \mathbf{B}\cdot(r\sin\theta dr\,d\phi\,\widehat{\mathbf{a}}_\theta) \\
&= \int B_\theta(r\sin\theta\,dr\,d\phi) = \int (r\cos\phi)(r\sin\theta\,dr\,d\phi) \\
&= \sin\theta\int_0^2 r^2dr\int_0^{2\pi}\cos\phi\,d\phi = \sin(\pi/2)(1/3)r^3\,|_0^2\,\sin\phi\,|_0^{2\pi} \\
&= 0
\end{aligned}$$

Thus,

$$\oint \mathbf{B}\cdot\mathbf{dS} = \int_{top} + \int_{bottom} = 0$$

so there is no net flux through the closed hemisphere.