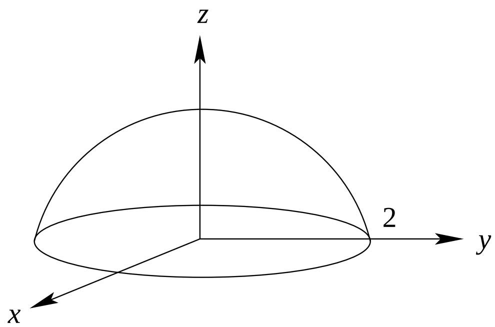
EE331—EXAMPLE #16: EXAMPLE #15 REVISITED

Given the vector field $\mathbf{B}(r,\theta,\phi) = r^2\cos\phi\,\widehat{\mathbf{a}}_r + r\cos\phi\,\widehat{\mathbf{a}}_\theta$, find the net outward flux through the closed hemisphere $0 \le r \le 2$, $0 \le \theta \le \pi/2$, and $0 \le \phi < 2\pi$ using the divergence theorem.

First make a sketch of the surface. (Still sorry about the strange perspective.)



In Example #15 we found the net flux through the closed surface by dividing the surface into two open surfaces, the top and bottom. However, according to the divergence theorem:

$$\oint \mathbf{B}\cdot\mathbf{dS} = \int \nabla\cdot\mathbf{B}\,dv \tag{1}$$

So let's take the divergence of $\mathbf{B}$:

$$\begin{aligned}
\nabla\cdot\mathbf{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(B_\theta\sin\theta) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial B_\phi}{\partial\phi}} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\cos\phi\sin\theta) \\
&= 4r^2\cos\phi + \frac{\cos\theta\cos\phi}{\sin\theta}
\end{aligned}$$

Using this result in (1) together with $dv = r^2\sin\theta dr d\theta d\phi$ gives:

$$\begin{aligned}
\int \nabla\cdot\mathbf{B}\,dv &= \int 4r^4\sin\theta\cos\phi dr d\theta d\phi + \int r^2\cos\theta\cos\phi dr d\theta d\phi \\
&= 4\int_0^2 r^4 dr\int_0^{\pi/2}\sin\theta d\theta \cancelto{0}{\int_0^{2\pi}\cos\phi d\phi} + \int_0^2 r^2 dr\int_0^{\pi/2}\cos\theta d\theta \cancelto{0}{\int_0^{2\pi}\cos\phi d\phi} \\
&= 0
\end{aligned} \tag{2}$$

which is the same result as we got in Example #15 using two surface integrals. Sometimes using the divergence theorem can make life easier (either by doing the volume integral instead of the surface integral or vice versa); sometimes it's a wash (as is probably the case here). Regardless, it gives you a way of checking your result!