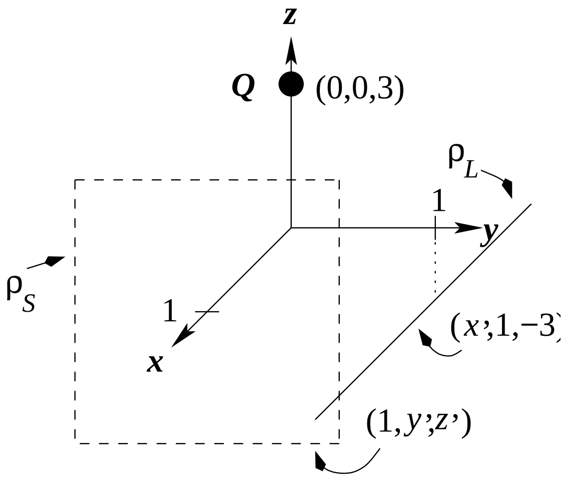
A point charge of 10 nC is located at (0,0,3). An infinite line $y = 1$, $z = -3$ carries a charge of 30 nC/m, and an infinite plane $x = 1$ carries a charge of 20 nC/m^2. Find **E** at the origin. Assume $\varepsilon = \varepsilon_0$.

First make a sketch using Cartesian coordinates. **E** is due to point, line, and surface charges:



$$\mathbf{E} = \mathbf{E}_Q + \mathbf{E}_L + \mathbf{E}_S \tag{1}$$

We find each **E** independently and then add the results together.

Point Charge:

For a point charge we have:

$$\mathbf{E}_Q = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2}$$

We can't exploit symmetry so we proceed to identify Q, $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}-\mathbf{r}'$, and $|\mathbf{r}-\mathbf{r}'|^3$. Q is the value of the point charge, $\mathbf{r}$ is the observation location (for this problem, the origin) where we want to calculate the value of the electric field, and $\mathbf{r}'$ is the location of the point source that gives rise to the electric field. Thus,

$$Q = 10^{-8},\ \ \mathbf{r} = (0,0,0),\ \ \mathbf{r}' = (0,0,3),\ \ \mathbf{r}-\mathbf{r}' = (0,0,-3)$$

$$|\mathbf{r}-\mathbf{r}'|^3 = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2} = \left[(-3)^2\right]^{3/2} = 27$$

[Note that we've suppressed units here, but always keep them in mind!] Using these values in (2) gives

$$\mathbf{E}_Q = \frac{10^{-8}}{4\pi\varepsilon_0}\frac{(0,0,-3)}{27} = -\frac{10^{-8}}{36\pi\varepsilon_0}\widehat{\mathbf{a}}_z = -9.9864\widehat{\mathbf{a}}_z\ \ [\mathrm{V/m}] \tag{3}$$

Line Charge:

The electric field equation for a line of charge is given by:

$$\mathbf{E}_L = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dl' \tag{4}$$

where the line charge density ρ_L can be a function of position. By symmetry, at the origin there is no E_x component (*explain!*). Again, we identify all the parameters needed to perform the integration:

$$\rho_L = 3\times 10^{-8},\ \ \mathbf{r} = (0,0,0),\ \ \mathbf{r}' = (x',1,-3),\ \ \mathbf{r}-\mathbf{r}' = (-x',-1,3)$$

$$|\mathbf{r}-\mathbf{r}'|^3 = \left[(-x')^2 + (-1)^2 + (3)^2\right]^{3/2} = [x'^2+10]^{3/2},\ \ dl' = dx'$$

and limits of integration from $-\infty$ to ∞. We use all these values in (4) to get:

$$\mathbf{E}_L \ = \ \frac{3\times 10^{-8}}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{(\cancelto{0}{-x'},-1,3)\,dx'}{[x'^2+10]^{3/2}} \tag{5}$$

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We can use an integral table to get:

$$\int_{-\infty}^{\infty} \frac{dx}{[x^2+a^2]^{3/2}} = 2\int_{0}^{\infty} \frac{dx}{[x^2+a^2]^{3/2}} = \frac{2x}{a^2\sqrt{x^2+a^2}}\Big|_0^\infty = \frac{2}{a^2}$$

We'll come across this integral often.

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Using this result in (5) gives:

$$\mathbf{E}_L = \frac{3\times 10^{-8}}{4\pi\varepsilon_0}(0,-1,3)\frac{2}{10} = -53.9265\widehat{\mathbf{a}}_y + 161.7794\widehat{\mathbf{a}}_z \ \ \mathrm{V/m} \tag{6}$$

Surface Charge:

The electric field equation for a surface of charge is given by:

$$\mathbf{E}_S = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} dS' \tag{7}$$

where, as with the line charge, the surface charge density ρ_S can be a function of position. By symmetry, at the origin there is no E_y or E_z component (*explain!*). Again, we identify all the parameters needed to perform the integration:

$$\rho_S = 2\times 10^{-8}, \ \ \mathbf{r} = (0,0,0), \ \ \mathbf{r}' = (1, y', z'), \ \ \mathbf{r}-\mathbf{r}' = (-1, -y', -z')$$

$$|\mathbf{r}-\mathbf{r}'|^3 = \left[(-1)^2 + (-y')^2 + (-z')^2\right]^{3/2} = [1 + y'^2 + z'^2]^{3/2}, \ \ dS' = dy'dz'$$

and limits of integration from $-\infty$ to ∞ for both integrals. We use all these values in (7) to get:

$$\begin{aligned}
\mathbf{E}_S &= \frac{2\times 10^{-8}}{4\pi\varepsilon_0}\int_{\infty}^{\infty} dy' \int_{-\infty}^{\infty} \frac{(-1, \cancelto{0}{-y'}, \cancelto{0}{-z'})}{[1+y'^2+z'^2]^{3/2}} dz' \\
&= -\frac{2\times 10^{-8}}{4\pi\varepsilon_0}\widehat{\mathbf{a}}_x \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} \frac{dz'}{[1+y'^2+z'^2]^{3/2}} \\
&= -\frac{2\times 10^{-8}}{4\pi\varepsilon_0}\widehat{\mathbf{a}}_x \int_{-\infty}^{\infty} \frac{2dy'}{y'^2+1} = -\frac{2\times 10^{-8}}{\pi\varepsilon_0}\widehat{\mathbf{a}}_x \int_{0}^{\infty} \frac{dy'}{y'^2+1} \\
&= -\frac{2\times 10^{-8}}{\pi\varepsilon_0}\widehat{\mathbf{a}}_x \tan^{-1} y'\Big|_0^\infty = -\frac{2\times 10^{-8}}{\pi\varepsilon_0}\widehat{\mathbf{a}}_x \frac{\pi}{2} = -1,129.4330\widehat{\mathbf{a}}_x \ \ \mathrm{V/m}
\end{aligned} \tag{8}$$

Finally, we add (3), (6), and (8) together to find the total electric field at the origin:

$$\mathbf{E}(0,0,0) = \mathbf{E}_Q + \mathbf{E}_L + \mathbf{E}_S = -1,129.4330\widehat{\mathbf{a}}_x - 53.9265\widehat{\mathbf{a}}_y + 151.7930\widehat{\mathbf{a}}_z \ \ \mathrm{V/m}$$