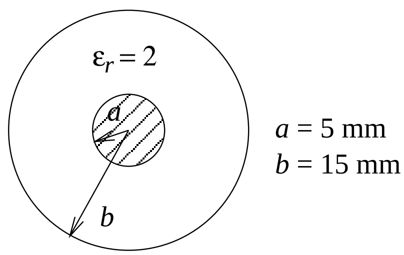
A cylindrical capacitor has inner and outer radii of $a = 5$ mm and $b = 15$ mm, respectively. If $V(a) = 100$ V and $V(b) = 0$ V, find $V$, $\mathbf{E}$, and $\mathbf{D}$ inside the capacitor. Next find $\rho_S$ on each plate and the value of the capacitance per unit length. The dielectric constant is 2.




BV: (1) $V$(5 mm) = 100 V
(2) $V$(15 mm) = 0 V

No charge density is given so we use Laplace's equation.

$$\nabla^2 V = 0$$

We choose cylindrical coordinates and note that, by symmetry, $V$ varies only as a function of $\rho$ (is this really true?). Thus, Laplace's equation becomes:

$$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

We integrate once with respect to $\rho$ and divide both sides by $\rho$ to get:

$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

Integrating again gives:

$$V(\rho) = A\ln(\rho) + B$$

We apply the values at the boundaries:

$$V(0.015) = A\ln(0.015) + B = 0 \quad \rightarrow \quad B = -A\ln(0.015)$$

$$V(0.005) = A\ln(0.005) - A\ln(0.015) = 100 \quad \rightarrow \quad A = -\frac{100}{\ln(3)}$$

Thus,

$$V(\rho) = -\frac{100}{\ln(3)}\ln\left(\frac{\rho}{0.015}\right)\ \mathrm{V}$$

We use the negative of the gradient of the potential to obtain $\mathbf{E}$ and find $\mathbf{D}$ from $\mathbf{E}$:

$$\mathbf{E}(\rho) = -\nabla V = -\frac{dV}{d\rho}\hat{\mathbf{a}}_\rho = \frac{100}{\ln(3)\rho}\hat{\mathbf{a}}_\rho\ \mathrm{V/m}$$

$$\mathbf{D}(\rho) = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E} = 2\varepsilon_0\mathbf{E} = \frac{200\varepsilon_0}{\ln(3)\rho}\hat{\mathbf{a}}_\rho\ \mathrm{C/m^2}$$

Next we use one of the boundary conditions for a dielectric-conductor interface to find the surface charge density $\rho_S$.

$$\rho_S(0.005) = D_\rho(0.005) = \frac{200\varepsilon_0}{\ln(3)(0.005)} = 322.3703\ \mathrm{nC/m^2}$$

$$\rho_S(0.015) = -\frac{200\varepsilon_0}{\ln(3)(0.015)} = -107.4568\mathrm{nC/m^2}$$

The charge on each conductor should have the same value but opposite signs:

$$Q^+ = \rho_S S = \rho_S(2\pi\rho L) = (322.3703 \times 10^{-9})(2\pi(0.005)L) = 10.1276L \text{ nC}$$

$$Q^- = (-107.4568 \times 10^{-9})(2\pi(0.015)L) = -10.1276L \text{ nC}$$

Finally, the capacitance per unit length is found from:

$$C = \frac{Q}{V} = \frac{10.1276 \times 10^{-9}L}{100} = 101.2760L \text{ pF}$$

or

$$\frac{C}{L} = 101.2760 \text{ pF/m}$$