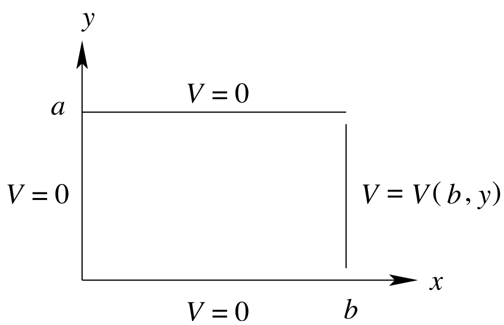
EE331—EXAMPLE #26: LAPLACE'S EQUATION, 2-D PROBLEM

Use the general solution to Laplace's equation for $V(0,y) = V(x,0) = V(x,a) = 0$ to find the complete solution to Laplace's equation when $V(b,y) = V_0 =$ constant.



The general solution is given by:

$$V(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n\pi}{a}y\right) \text{ V}$$

Apply the non-zero boundary value:

$$V(b,y) = V_0 = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a}y\right)$$

Multiply both sides by $\sin\left(\frac{m\pi}{a}y\right)$ and integrate:

$$\begin{aligned}
\int_0^a V_0 \sin\left(\frac{m\pi}{a}y\right) dy &= \int_0^a \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a}y\right) \sin\left(\frac{m\pi}{a}y\right) dy \\
&= \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi b}{a}\right) \int_0^a \sin\left(\frac{n\pi}{a}y\right) \sin\left(\frac{m\pi}{a}y\right) dy
\end{aligned}$$

Look at the right-hand side of the equation. The only non-zero term occurs when $m = n$. Thus:

$$RHS = c_n \sinh\left(\frac{n\pi b}{a}\right)\left(\frac{a}{2}\right) \tag{1}$$

Now integrate the left-hand side to get:

$$LHS = -V_0 \frac{a}{n\pi} \cos\left(\frac{n\pi}{a}y\right) |_0^a = V_0 \frac{a}{n\pi}(1 - \cos(n\pi)) \tag{2}$$

Now consider (2). When n is an even integer, $\cos(n\pi) = 1$, and when n is an odd integer, $\cos(n\pi) = -1$. So (2) is 0 for even n and for n odd:

$$LHS = V_0 \frac{2a}{n\pi} \tag{3}$$

Equating (1) and (3) gives:

$$c_n \sinh\left(\frac{n\pi b}{a}\right) \frac{a}{2} = V_0 \frac{2a}{n\pi}$$

Thus, the coefficients associated with the sine terms are given by:

$$c_n = \frac{4V_0}{\pi} \frac{1}{n \sinh(\frac{n\pi b}{a})}$$

and the complete solution for our problem (i.e., the value of the potential everywhere inside the region of interest) is:

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n \ odd} \frac{\sinh(\frac{n\pi}{a}x) \sin(\frac{n\pi}{a}y)}{n \sinh(\frac{n\pi b}{a})} \text{ V}$$