The electric field in a region of space is $\mathbf{E} = -4\hat{\mathbf{a}}_y$ V/m and the magnetic flux density is $\mathbf{B} = 5\hat{\mathbf{a}}_x$ Wb/m^2. If a particle with mass 1 kg and charge 1 C starts from rest at point (2,−3,4), calculate (a) its velocity at $t = 1$ s, (b) its location at $t = 1$ s, and (c) its kinetic energy at $t = 1$ s.

(a) Equate the Lorentz force equation and Newton's second law of motion:

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = m\mathbf{a} = m\frac{d\mathbf{u}}{dt} \tag{1}$$

Q, $\mathbf{E}$, $\mathbf{B}$, and m are given; $\mathbf{u}$ is the unknown velocity we want to find. First we find the cross product $\mathbf{u} \times \mathbf{B}$:

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ u_x & u_y & u_z \\ 5 & 0 & 0 \end{vmatrix} = 5u_z\hat{\mathbf{a}}_y - 5u_y\hat{\mathbf{a}}_z$$

We use this together with the quantities given in (1) to get:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \frac{Q}{m}(-4\hat{\mathbf{a}}_y + 5u_z\hat{\mathbf{a}}_y - 5u_y\hat{\mathbf{a}}_z) \\ &= \frac{1}{1}(-4\hat{\mathbf{a}}_y + 5u_z\hat{\mathbf{a}}_y - 5u_y\hat{\mathbf{a}}_z) \end{aligned}$$

This is a vector differential equation which is equivalent to three scalar differential equations:

$$\frac{du_x}{dt} = 0 \tag{2}$$

$$\frac{du_y}{dt} = 5u_z - 4 \tag{3}$$

$$\frac{du_z}{dt} = -5u_y \tag{4}$$

Equations (3) and (4) are coupled differential equations. We can decouple them by taking the second derivative with respect to t of (3) to get:

$$\frac{d^2u_y}{dt^2} = 5\frac{du_z}{dt} \tag{5}$$

Now use (4) in (5):

$$\frac{d^2u_y}{dt^2} = 5(-5u_y) = -25u_y \quad \rightarrow \quad \frac{d^2u_y}{dt^2} + 25u_y = 0 \tag{6}$$

We solve (6) to get:

$$u_y(t) = A\cos(5t) + B\sin(5t) \tag{7}$$

Now we need initial conditions to find the constants in Eq. (7). We know $u_x = u_y = u_z = 0$ (why?). Use $u_y = 0$ in (7):

$$u_y(0) = A = 0$$

Thus:

$$u_y(t) = B\sin(5t) \tag{8}$$

Now use (8) in (3) and solve for u_z:

$$\frac{du_y}{dt} = 5B\cos(5t) = 5u_z - 4$$

$$u_z(t) = B\cos(5t) + \frac{4}{5} \tag{9}$$

Next apply the initial condition $u_z(0) = 0$:

$$u_z(0) = B + \frac{4}{5} = 0 \quad \rightarrow \quad B = -\frac{4}{5}$$

Thus, from (8) and (9):

$$u_y(t) = -\frac{4}{5}\sin(5t) \tag{10}$$

$$u_z(t) = -\frac{4}{5}\cos(5t) + \frac{4}{5} \tag{11}$$

Next, solve (2):

$$u_x(t) = C$$

And applying the initial condition $u_x(0) = 0$ gives:

$$u_x(0) = C = 0 \quad \rightarrow \quad u_x(t) = 0$$

This equation and (10) and (11) give us the solution to (1):

$$\mathbf{u}(t) = \left(-\frac{4}{5}\sin(5t)\right)\hat{\mathbf{a}}_y - \left(\frac{4}{5}\cos(5t) - \frac{4}{5}\right)\hat{\mathbf{a}}_z \text{ m/s} \tag{12}$$

At $t = 1$ s,

$$\boxed{\mathbf{u}(1) = (0, 0.7671, 0.5731) \text{ m/s}}$$

(b) Recall that:

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}$$

and $\mathbf{r} = (x, y, z)$. Then from (12), we get:

$$\frac{dx}{dt} = 0 \tag{13}$$

$$\frac{dy}{dt} = -\frac{4}{5}\sin(5t) \tag{14}$$

$$\frac{dz}{dt} = -\frac{4}{5}\cos(5t) + \frac{4}{5} \tag{15}$$

Integrate (13)-(15) to get:

$$x = A$$

$$y = \frac{4}{25}\cos(5t) + B$$

$$z = -\frac{4}{25}\sin(5t) + \frac{4}{5}t + C$$

The initial conditions for the location are $x = 2$, $y = -3$, and $z = 4$. Applying these gives:

$$x(0) = A = 2$$

$$y(0) = \frac{4}{25} + B = -3 \quad \rightarrow \quad B = -\frac{79}{25}$$

$$z(0) = C = 4$$

Thus:

$$\mathbf{r}(t) = 2\hat{\mathbf{a}}_x + \left(\frac{4}{25}\cos(5t) - \frac{79}{25}\right)\hat{\mathbf{a}}_y + \left(-\frac{4}{25}\sin(5t) + \frac{4}{5}t + 4\right)\hat{\mathbf{a}}_z \text{ m}$$

At $t = 1$ s,

$$\boxed{\mathbf{r}(1) = (2, -3.1146, 4.9534) \text{ m}}$$

(c) The kinetic energy at $t = 1$ s is

$$\boxed{KE = \frac{1}{2}mu^2 = \frac{1}{2}(1)[(0.7671)^2 + (0.5731)^2] = 0.4585 \text{ J}}$$