EE 331—Solution to the Scalar Wave Equation

The scalar wave equation is a second-order PDE given by:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

The general solution to this PDE is:

$$y(x,t) = y^+(t - x/u) + y^-(t + x/u) \tag{2}$$

which is the sum of a wave traveling in the positive x direction and a wave traveling in the negative x direction. How do we know $y(x,t)$ is the solution? Substitute it into both sides of Eq. (1). If LHS = RHS, then we've shown it's the solution. Consider only the wave traveling in the positive x direction, $y^+(x,t)$. Let

$$\tau = t - x/u,$$

then

$$\frac{\partial \tau}{\partial t} = 1 \tag{3}$$

and

$$\frac{\partial \tau}{\partial x} = -1/u \tag{4}$$

Consider the LHS of Eq. (1). Take the first derivative of y^+ wrt x:

$$\frac{\partial y^+}{\partial x} = \frac{\partial y^+}{\partial \tau}\frac{\partial \tau}{\partial x} = -\frac{1}{u}\frac{\partial y^+}{\partial \tau} \tag{5}$$

using the chain rule and Eq. (4). Now take the second derivative of y^+ wrt x:

$$\frac{\partial^2 y^+}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y^+}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{u}\frac{\partial y^+}{\partial x}\right) = \frac{\partial}{\partial \tau}\left(\frac{1}{u^2}\frac{\partial y^+}{\partial \tau}\right) = -\frac{1}{u^2}\frac{\partial^2 y^+}{\partial \tau^2} \tag{6}$$

where we've again used the chain rule and have also changed the order of differentiation. Similarly, we consider the RHS of Eq. (1). We take the first and second derivatives of y^+ wrt t, and we end up getting

$$\frac{\partial^2 y^+}{\partial t^2} = \frac{\partial^2 y^+}{\partial \tau^2} \tag{7}$$

If we multiply both sides of Eq. (7) by $1/u^2$, we see that the LHS and RHS of Eq. (1) are equal. We can repeat this exercise to show that y^- is also a solution to Eq. (1).