EE 331—Notes on the Taylor Series Expansion

Recall from Math 172 (calculus) that a Taylor series expansion of a function $f(x)$ about a point x_0 can be used to approximate the value of the function near the point. This comes in handy and is used a lot.

A Taylor series is an infinite series, but generally only a few terms of the series are retained. It's written as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + f'''(x_0)\frac{(x - x_0)^3}{3!} + f''''(x_0)\frac{(x - x_0)^4}{4!} + \cdots$$

For example, consider $f(x) = \sin(x)$ expanded about $x_0 = 0$:

$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \end{aligned}$$

and so on. The first two non-zero terms give us:

$$\sin(x) = x - \frac{x^3}{6} + \cdots \approx x - \frac{x^3}{6} \tag{1}$$

So suppose we want to approximate $\sin(x)$ for $x = 0.5$. From (1),

$$\sin(0.5) \approx 0.5 - \frac{(0.5)^3}{6} = 0.4792$$

Using my HP calculator, I get $\sin(0.5) = 0.4794$. Thus, using just the first two non-zero terms of the Taylor series for the sine function, I'm able to find the value of $\sin(0.5)$ to 3 decimal places of accuracy!

Note that:

1. Eq. (1) is a polynomial series, and a polynomial series is easy to handle (e.g., it's easy to integrate a polynomial series). Since it's a polynomial series, the Taylor series approximates a function as the sum of a constant (x^0), a line (x^1), a parabola (x^2), and so on.

2. The first couple of terms of a Taylor series approximates the function well only when $\Delta x = x - x_0$ is small. For example, $\sin(\pi) \approx \pi - \frac{(\pi)^3}{6} = -2.0261$, but we know $\sin(\pi) = 0$. You have to think about what you're doing!

3. Finally, if a function is odd or even, its Taylor series must be odd or even.