Name ______________________________

EE 331 – Exam #3
Apr. 17, 2020
Closed Book/Closed Notes

Relax and think carefully. Show all your work to insure maximum partial credit. Remember that *neatness counts*! **SHOW ALL UNITS (e.g., V/m, C), and COMMUNICATE WHAT YOU KNOW!!**

1. (21 pts—1.5 pts each) Match the one term on the right that is **BEST** associated with the term on the left. *Each term on the right can only be used once, and not all terms on the right are used.* **No need to write down the terms on the left; just write 1-14, and put the letter next to it.**

1)	B	polarization density	A.	$\nabla^2 V = 0$
2)	G	Joule's law	B.	$\mathbf{P}$
3)	K	use to find electric field	C.	$\mathbf{J} = \sigma \mathbf{E}$
4)	L	relaxation time	D.	$W = -Q \int \mathbf{E} \cdot \mathbf{dl}$
5)	I	fundamental problem in EM	E.	$\varepsilon(\mathbf{r}) = \varepsilon$
6)	E	linear, isotropic, homogeneous	F.	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
7)	F	Poisson's equation	G.	$P = \int_v \mathbf{J} \cdot \mathbf{E}$
8)	S	law of superposition	H.	χ_e
9)	D	work	I.	charge here, effect there
10)	N	static electric fields are conservative	J.	$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$
11)	C	Ohm's law in point form	K.	$-\nabla V$
12)	H	electric susceptibility	L.	$T_r = \frac{\varepsilon}{\sigma}$
13)	J	continuity of current	M.	ε_r
14)	M	dielectric constant	N.	$\nabla \times \mathbf{E} = 0$
			O.	$\nabla \cdot \mathbf{D} = \rho_v$
			P.	ε_0
			S.	$\mathbf{F} = \sum_{k=1}^{N} \mathbf{F}_k$

2. (a) (8 pts) If the electric flux density $\mathbf{D}$ is $2\rho \sin\phi \widehat{\mathbf{a}}_\rho + \rho\cos\phi \widehat{\mathbf{a}}_\phi$ C/m^2, find the volume charge density ρ_v.

Use Gauss's law (M's first): $\nabla\cdot\mathbf{D} = \rho_v$

$$\begin{aligned}\nabla\cdot\mathbf{D} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(2\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho\cos\phi)\\ &= \frac{1}{\cancel{\rho}}(4\cancel{\rho}\sin\phi) - \frac{1}{\cancel{\rho}}(\cancel{\rho}\sin\phi)\\ &= \boxed{3\sin\phi \ \text{C/m}^3}\end{aligned}$$

(b) (8 pts) If the dipole moment $p = Qd = 4\pi\varepsilon_0$, what is the electric field in the $z = 0$ plane at $r = 1$ m due to an electric dipole in free space?

$$\mathbf{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\widehat{\mathbf{a}}_r + \sin\theta\,\widehat{\mathbf{a}}_\theta) = \frac{\cancel{4\pi\varepsilon_0}}{\cancel{4\pi\varepsilon_0}(1)^3}(2\cos(\pi/2)\,\widehat{\mathbf{a}}_r + \sin(\pi/2)\,\widehat{\mathbf{a}}_\theta) = \boxed{\widehat{\mathbf{a}}_\theta \ \text{V/m}}$$

(c) (12 pts) Region 1 ($z > 0$) has a dielectric constant of 3. Region 2 ($z < 0$) has a dielectric constant of 1.5. If $\mathbf{D}_2 = 12\widehat{\mathbf{a}}_x + 9\widehat{\mathbf{a}}_z$, find $\mathbf{D}_1$.

We're given:

$$\mathbf{D}_2 = 12\widehat{\mathbf{a}}_x + 9\widehat{\mathbf{a}}_z \ \text{C/m}^2$$

from which we get:

$$\mathbf{D}_{2t} = 12\widehat{\mathbf{a}}_x, \quad \mathbf{D}_{2n} = 9\widehat{\mathbf{a}}_z$$

The BC are given by:

BC: 1) $E_{1t} = E_{2t}$
2) $D_{1n} = D_{2n}$

From the first BC:

$$E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2} = \frac{D_{1t}}{\varepsilon_1}$$

$$\rightarrow D_{1t} = \frac{\varepsilon_1}{\varepsilon_2}D_{2t} = \frac{3\varepsilon_0}{1.5\varepsilon_0}(12) = 24$$

From the second BC:

$$D_{1n} = D_{2n} = 9$$

Thus:

$$\boxed{\mathbf{D}_1 = 24\widehat{\mathbf{a}}_x + 9\widehat{\mathbf{a}}_z \ \text{C/m}^2}$$

3. Two infinite conducting sheets are placed in free space with their edges meeting along the z axis but not touching. One sheet is in the plane $\phi = 0$ and has $V = 0$, and the other sheet is in the plane $\phi = \pi/4$ and has $V = 25$ V.

   (a) (16 pts) Find the potential in the region between the two conductors.

   Start with Laplace's equation:

$$\nabla^2 V = 0 \quad \rightarrow \quad \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0 \quad \rightarrow \quad \frac{d^2V}{d\phi^2} = 0$$

   Next integrate twice wrt ϕ:

$$\begin{aligned} \frac{dV}{d\phi} &= A \\ V(\phi) &= A\phi + B \end{aligned}$$

   Next apply the values at the boundaries:

$$\begin{aligned} V(0) &= A(0) + B \quad \rightarrow \quad B = 0 \\ V(\pi/4) &= A\left(\frac{\pi}{4}\right) = 25 \quad \rightarrow \quad A = \frac{100}{\pi} \end{aligned}$$

   Finally, then:

$$\boxed{V(\phi) = \frac{100}{\pi}\,\phi \text{ V}}$$

   (b) (10 pts) Using the uniqueness theorem, prove that your solution is the correct and only solution. **Hint: You must show two things to satisfy the uniqueness theorem! If you're unable to solve part (a), then explain these two things to get most of the credit.**

   First show that V is a solution to Laplace's equation:

$$\frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = \frac{1}{\rho^2}\frac{d^2}{d\phi^2}\left(\frac{100}{\pi}\,\phi\right) = \frac{1}{\rho^2}\frac{d}{d\phi}\left(\frac{100}{\pi}\right) = 0$$

   Next show V satisfies the values at the boundaries:

$$V(0) = \frac{100}{\pi}(0) = 0$$

$$V(\pi/4) = \frac{\cancelto{25}{100}}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{\cancel{4}}\right) = 25 \text{ V}$$

4. An infinite line $(0, -2, z)$ carries a charge $\rho_L = 2$ C/m in free space. Find the value of the electric field at the origin using Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_L(\mathbf{r} - \mathbf{r}\,')}{|\mathbf{r} - \mathbf{r}\,'|^3} dl\,' \ \ [\mathrm{V/m}] \tag{1}$$

(a) (7 pts) What are $\mathbf{r}$, $\mathbf{r}\,'$, $\mathbf{r} - \mathbf{r}\,'$, and $|\mathbf{r} - \mathbf{r}\,'|^3$?

$$\mathbf{r} = 0, \quad \mathbf{r}' = (0, -2, z') \ \ \mathrm{m}$$

$$\mathbf{r} - \mathbf{r}' = (0, 2, -z') \ \ \mathrm{m}, \quad |\mathbf{r} - \mathbf{r}'|^3 = [z'^2 + 4]^{3/2} \ \ \mathrm{m}^3$$

(b) (4 pts) What are $dl\,'$ and the limits of integration?

$$dl' = dz', \quad \int_{-\infty}^{\infty} dz'$$

(c) (4 pts) By symmetry, what component(s) of $\mathbf{E}$, if any, will be zero (it might help to make a sketch)?

$$E_x, \ \ E_z$$

(d) (10 pts) Now use all this information in Eq. (1) to find $\mathbf{E}$ at the origin. Leave your answer in terms of ε_0. Make sure your answer makes physical sense! *Some integrals are given on the equation sheet.*

$$\begin{aligned}
\mathbf{E}(0) &= \frac{2}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{(\overset{0}{\cancel{0}}, 2, \overset{0}{\cancel{-z'}})}{[z'^2 + 4]^{3/2}} dz' \\
&= \frac{2}{\pi\varepsilon_0} \widehat{\mathbf{a}}_y \int_0^{\infty} \frac{dz'}{[z'^2 + 4]^{3/2}} \quad \text{(even function)} \\
&= \frac{2}{\pi\varepsilon_0} \widehat{\mathbf{a}}_y \left(\frac{z'}{4\sqrt{z'^2 + 4}} \Big|_0^{\infty} \right) \quad \text{(from equation sheet)} \\
&= \boxed{\frac{1}{2\pi\varepsilon_0} \widehat{\mathbf{a}}_y \ \ \mathrm{V/m}}
\end{aligned}$$