Name ____________________

EE 331 – Exam #4
May 6, 2020
Closed Book/Closed Notes

Relax and think carefully. Show all your work to insure maximum partial credit. Remember that *neatness counts*! **SHOW ALL UNITS (e.g., A/m, Wb), and COMMUNICATE WHAT YOU KNOW!!**

1. (a) (10 pts) A circle of wire of radius 1 m in the $x = 0$ plane carries a current of 2 A in the counter-clockwise direction. If the wire is in a uniform magnetic field with $\mathbf{B} = \frac{4}{\pi}\,\widehat{\mathbf{a}}_y$ Wb/m^2, find the torque on it.

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = IS\,\widehat{\mathbf{a}}_n \times \mathbf{B} = I\pi r^2\,\widehat{\mathbf{a}}_n \times \mathbf{B} = 2\pi\,\widehat{\mathbf{a}}_x \times \frac{4}{\pi}\,\widehat{\mathbf{a}}_y = \boxed{8\,\widehat{\mathbf{a}}_z\ \mathrm{N \cdot m}}$$

1. (b) (10 pts) Three straight conductors L_1, L_2, and L_3 are lined up left to right parallel to the z-axis. L_1 and L_2 have currents of 1 A in the $+\widehat{\mathbf{a}}_z$ direction, and L_3 has a current of 2 A in the $+\widehat{\mathbf{a}}_z$ direction. If the distance between L_1 and L_2 is 1 m, what must the distance be between L_2 and L_3 to make the net force on L_2 zero? If the net force can't be zero, explain why. Recall that for a straight conductor, $\mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\,\widehat{\mathbf{a}}_\phi$. *A sketch will probably help you!*

Because all currents are in the same direction, L_2 will be attracted to both L_1 and L_3. Thus:

$$F_2 = I_2 L_2 B_1 - I_2 L_2 B_3 = 0 \quad \rightarrow \quad B_1 = B_3$$

$$\rightarrow \quad \frac{\mu_0 I_1}{2\pi\rho_1} = \frac{\mu_0 I_3}{2\pi\rho_3}$$

$$\rightarrow \quad \frac{1}{1} = \frac{2}{\rho_3}$$

$$\rightarrow \quad \boxed{\rho_3 = 2\ \mathrm{m}}$$

1. (c) (5 pts) A circle of radius a encircles a current of 2 A. Find $\oint \mathbf{H}\cdot\mathbf{dl}$ around this circle. Note: This problem isn't impossible; it's actually trivial if you just look at the equations sheet and think a little.

By Ampere's law:

$$\boxed{\oint \mathbf{H}\cdot\mathbf{dl} = I_{enc} = 2\ \mathrm{A}}$$

2. (a) (7 pts) For a magnetic field $\mathbf{H}$ of $\frac{2z}{\rho}\,\widehat{\mathbf{a}}_\rho + \ln\rho\,\widehat{\mathbf{a}}_z$ A/m, find the volume current density $\mathbf{J}$.

Use Amp's law: $\nabla \times \mathbf{H} = \mathbf{J}$

$$\begin{aligned}
\nabla \times \mathbf{H} &= \left[\frac{\partial}{\partial z}\left(\frac{2z}{\rho}\right) - \frac{\partial}{\partial \rho}(\ln\,\rho)\right]\widehat{\mathbf{a}}_\phi \\
&= \left[\frac{2}{\rho} - \frac{1}{\rho}\right]\widehat{\mathbf{a}}_\phi \\
&= \boxed{\frac{1}{\rho}\widehat{\mathbf{a}}_\phi \;\; \mathrm{A/m^2}}
\end{aligned}$$

2. (b) (7 pts) If the magnitude of the magnetic moment is $m = 4\pi$ A-m^2, what is the magnetic field in the $z = 0$ plane at $r = 1$ m due to a magnetic dipole in free space?

$$\mathbf{H} = \frac{m}{4\pi r^3}(2\cos\theta\,\widehat{\mathbf{a}}_r + \sin\theta\,\widehat{\mathbf{a}}_\theta) = \frac{\cancel{4\pi}}{\cancel{4\pi}(1)^3}(2\overbrace{\cancel{\cos(\pi/2)}}^{0}\widehat{\mathbf{a}}_r + \overbrace{\cancel{\sin(\pi/2)}}^{1}\widehat{\mathbf{a}}_\theta) = \boxed{\widehat{\mathbf{a}}_\theta \;\; \mathrm{A/m}}$$

2. (c) (11 pts) Region 1 ($z > 0$) has a relative permeability of of 3. Region 2 ($z < 0$) has a relative permeability of 1.5. If $\mathbf{B}_2 = 12\,\widehat{\mathbf{a}}_x + 9\,\widehat{\mathbf{a}}_z$ Wb/m^2, find $\mathbf{B}_1$. *We're given:*

$$\mathbf{B}_2 = 12\widehat{\mathbf{a}}_x + 9\widehat{\mathbf{a}}_z \;\; \mathrm{Wb/m^2}$$

from which we get:

$$\mathbf{B}_{2t} = 12\widehat{\mathbf{a}}_x, \quad \mathbf{B}_{2n} = 9\widehat{\mathbf{a}}_z$$

The BC are given by:

BC: 1) $H_{1t} = H_{2t}$
2) $B_{1n} = B_{2n}$

From the first BC:

$$H_{1t} = H_{2t} = \frac{B_{2t}}{\mu_2} = \frac{B_{1t}}{\mu_1}$$

$$\rightarrow B_{1t} = \frac{\mu_1}{\mu_2}B_{2t} = \frac{3\mu_0}{1.5\mu_0}(12) = 24$$

From the second BC:

$$B_{1n} = B_{2n} = 9$$

Thus:

$$\boxed{\mathbf{B}_1 = 24\widehat{\mathbf{a}}_x + 9\widehat{\mathbf{a}}_z \;\; \mathrm{Wb/m^2}}$$

3. A charged particle starts from rest at the origin in an electric field $\mathbf{E} = 2\,\widehat{\mathbf{a}}_y + 3\,\widehat{\mathbf{a}}_z$ V/m and a magnetic flux density $\mathbf{B} = 4\,\widehat{\mathbf{a}}_x$ Wb/m^2. The particle's mass is 1 kg, and its charge is 1 C.

   (a) (4 pts) What is the name of the EM equation that is used to solve this problem? Write it down.

$$\boxed{\text{Lorentz Force Equation : } \;\; \mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \;\; [N]}$$

   (b) (4 pts) Write down the two initial conditions.

$$\boxed{\mathbf{u}_0 = (0,0,0) \;\text{ and }\; \mathbf{r}_0 = (0,0,0)}$$

   (c) (5 pts) At time $t = 10$ s, is the kinetic energy of the particle the same as it was initially? Explain. **Note: Do not find the kinetic energy to answer this question.**

   Initially, $KE = 0$, but the applied electric field will cause the charged particle to move so that the kinetic energy will no longer be zero.

   (d) (12 pts) Find the three first-order differential equations needed to solve for the velocity of the particle. **NOTE: Do not solve these equations!**

   First we equate the LFE and Newton's second law of motion:

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = m\frac{d\mathbf{u}}{dt} \tag{1}$$

   Then we find $\mathbf{u} \times \mathbf{B}$:

$$\mathbf{u} \times \mathbf{B} = \begin{vmatrix} \widehat{\mathbf{a}}_x & \widehat{\mathbf{a}}_y & \widehat{\mathbf{a}}_z \\ u_x & u_y & u_z \\ 4 & 0 & 0 \end{vmatrix} = 4u_z\,\widehat{\mathbf{a}}_y - 4u_y\,\widehat{\mathbf{a}}_z$$

   We use this result in Eq. (1) and divide both sides by m to get:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \frac{Q}{m}\left(2\,\widehat{\mathbf{a}}_y + 3\,\widehat{\mathbf{a}}_z + 4u_z\,\widehat{\mathbf{a}}_y - 4u_y\,\widehat{\mathbf{a}}_z\right) \\ &= (4u_z + 2)\,\widehat{\mathbf{a}}_y + (3 - 4u_y)\,\widehat{\mathbf{a}}_z \end{aligned}$$

   From this we obtain our three first-order differential equations:

$$\boxed{\frac{du_x}{dt} = 0, \quad \frac{du_y}{dt} = 4u_z + 2, \quad \frac{du_z}{dt} = 3 - 4u_y}$$

4. A loop of current with a radius of 2 m in the $z = 0$ plane carries a current $I = 2$ A counter-clockwise looking down. Find the value of the magnetic field along the z-axis using the Biot-Savart law:

$$\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{I\mathbf{dl}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \ \ [\mathrm{A/m}] \tag{2}$$

(a) (7 pts) What are $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r} - \mathbf{r}'$, and $|\mathbf{r} - \mathbf{r}'|^3$?

$$\mathbf{r} = z\,\widehat{\mathbf{a}}_z \ \mathrm{m}, \quad \mathbf{r}' = 2\,\widehat{\mathbf{a}}_\rho \ \mathrm{m}$$

$$\mathbf{r} - \mathbf{r}' = z\,\widehat{\mathbf{a}}_z - 2\,\widehat{\mathbf{a}}_\rho \ \mathrm{m}, \quad |\mathbf{r} - \mathbf{r}'|^3 = [z^2 + 4]^{3/2} \ \mathrm{m}^3$$

(b) (6 pts) What are $\mathbf{dl}'$, $\mathbf{dl}' \times (\mathbf{r} - \mathbf{r}')$, and the limits of integration?

$$\mathbf{dl}' = \rho' d\phi'\,\widehat{\mathbf{a}}_\phi = 2d\phi'\,\widehat{\mathbf{a}}_\phi$$

$$\mathbf{dl}' \times (\mathbf{r} - \mathbf{r}') = 2d\phi'\,\widehat{\mathbf{a}}_\phi \times (z\,\widehat{\mathbf{a}}_z - 2\,\widehat{\mathbf{a}}_\rho) = (2z\,\widehat{\mathbf{a}}_\rho + 4\,\widehat{\mathbf{a}}_z)d\phi'$$

$$\int_0^{2\pi} d\phi'$$

(c) (2 pts) By symmetry, what component(s) of $\mathbf{H}$, if any, will be zero?

$$H_\rho$$

(d) (10 pts) Now use all this information in Eq. (2) to find $\mathbf{H}$ along the z-axis.

$$\begin{aligned} \mathbf{H}(z) &= \frac{2}{4\pi} \int_0^{2\pi} \frac{\cancelto{0}{2z\,\widehat{\mathbf{a}}_\rho} + 4\,\widehat{\mathbf{a}}_z}{[z^2 + 4]^{3/2}}\, d\phi' \\ &= \frac{2}{4\pi} \frac{4}{[z^2 + 4]^{3/2}}\,\widehat{\mathbf{a}}_z \int_0^{2\pi} d\phi' \\ &= \boxed{\frac{4}{[z^2 + 4]^{3/2}}\,\widehat{\mathbf{a}}_z \ \mathrm{A/m}} \end{aligned}$$