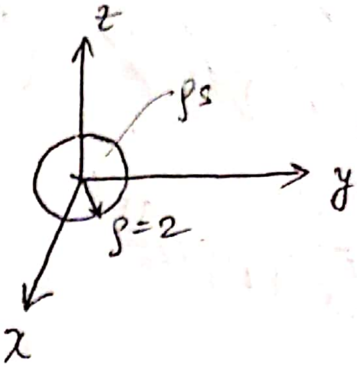
1.) A circular area of charge in free space with a radius of 2m is centered at the origin in the $z=0$ plane. It carries a charge of $\rho\ \mu C/m^2$



$\bar{r} = (0,0,z)\ m$

$\bar{r}' = \rho'\hat{a}_\rho\ .\ m$

$\bar{r} - \bar{r}' = -\rho'\hat{a}_\rho + z\hat{a}_z$

$|\bar{r} - \bar{r}'|^3 = [\rho'^2 + z^2]^{3/2}\ m^3.$

Differential surface $ds' = \rho' d\rho' d\phi'\ m^2.$

$\rho_s = \rho'\ \mu C/m^2$
$= \rho' \times 10^{-6}\ C/m^2$

$$\bar{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_s(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\,ds'$$

0 (due to symmetry)

$$= \frac{10^{-6}}{4\pi\epsilon_0}\int_{\phi'=0}^{\phi'=2\pi}\int_{\rho=0}^{\rho=2} \frac{\rho'(-\cancel{\rho'\hat{a}_\rho} + z\hat{a}_z)}{(\rho'^2+z^2)^{3/2}}\,\rho' d\rho' d\phi'.$$

$$= \frac{10^{-6}}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^2 \frac{\rho'^2 . z\hat{a}_z}{(\rho'^2+z^2)^{3/2}}\,d\rho'$$

$$= \frac{10^{-6} z\hat{a}_z}{4\pi\epsilon_0}\ \phi'\Big|_0^{2\pi}\int_0^2 \frac{\rho'^2 d\rho'}{(\rho'^2+z^2)^{3/2}}.$$

Now,

$$\cancel{\int \frac{x^2}{(x^2+a^2)^{3/2}}dx = \ln\left(|a\sqrt{x^2+a^2}+|a|x|\right) - \frac{x}{a\sqrt{x^2/a^2+1}}}$$

$$\cancel{\int \frac{\rho'^2\, d\rho}{(\rho'^2+z^2)^{3/2}} = \ln\left(|z|\sqrt{\rho'^2+z^2}+z\rho'\right)}$$

$$I = \int \frac{\rho'^2}{(\rho'^2+z^2)^{3/2}}d\rho'$$

substitute $\rho' = z\tan u$.

$u = \tan^{-1}(\rho/z)$.

$\therefore d\rho' = \sec^2 u . z\, du.$

$$= \int \frac{z^2\tan^2 u . \sec^2 u . z\, du}{z^3\sec^3 u}$$

$$= \int \frac{\tan^2 u}{\sec u}du.$$

$$= \int \cos u \tan^2 u\, du.$$

$$= \int \cos u(\sec^2 u - 1)\, du$$

$$= \int (\sec u - \cos u)\, du.$$

$$= \ln(\tan u + \sec u) - \sin u$$

$$= \ln\left[\rho'/z + \sqrt{1+\rho'^2/z^2}\right] - \frac{\rho'}{\sqrt{1+\rho'^2/z^2}}$$

$\tan u = \rho'/z.$

$\sec u = \sqrt{1+\rho'^2/z^2}$

$\cos u = \dfrac{1}{\sqrt{1+\rho'^2/z^2}}$

$\sin u = \sqrt{1 - \dfrac{1}{1+\rho'^2/z^2}}$

$= \dfrac{\rho'}{\sqrt{1+\rho'^2/z^2}}$

$$\int_0^2 \frac{\rho'^2 d\rho'}{(\rho'^2+z^2)^{3/2}} = \ln\left|\rho'/z + \sqrt{1+\rho'^2/z^2}\right|_{\rho'=0}^{\rho'=2} - \frac{\rho'}{\sqrt{1+\rho'^2/z^2}}\Bigg|_{\rho'=0}^{\rho'=2}.$$

$$= \ln\left|2/z + \sqrt{1+4/z^2}\right| - \ln|1| - \frac{2}{\sqrt{1+4/z^2}} - 0.$$

$$= \ln\left|2/z + \sqrt{1+4/z^2}\right| - \frac{2}{\sqrt{1+4/z^2}}$$

$$\bar{E} = \frac{10^{-6} z\, \hat{a}_z}{4\pi\epsilon_0} \cdot 2\pi \cdot \left[\ln\left|\frac{2}{z} + \sqrt{1+4/z^2}\right| - \frac{2}{\sqrt{1+4/z^2}}\right]$$

$$= \cancel{\frac{10^{-6}}{2\epsilon_0}\left[\ln\left|2 + \sqrt{z^2+4}\right|\right]}$$

$$= \frac{10^{-6} z}{2\epsilon_0}\left[\ln \frac{2+\sqrt{z^2+4}}{z} - \frac{2z}{\sqrt{z^2+4}}\right] \hat{a}_z$$

2) Electric field is given by

$\bar{D} = 2\rho(z+1)\cos\phi\, \hat{a}_\rho - \rho(z+1)\sin\phi\, \hat{a}_\phi + \rho^2\cos\phi\, \hat{a}_z$ $\mu C/m^2$

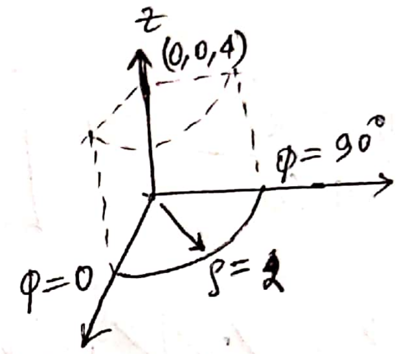
Charge density $\rho_v = \nabla\cdot\bar{D}$

$$\nabla\cdot\bar{D} = \frac{1}{\rho}\frac{d}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(2\rho^2(z+1)\cos\phi\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(-\rho(z+1)\sin\phi\right) + \frac{\partial}{\partial z}(\rho^2\cos\phi)$$

$$= 4(z+1)\cos\phi - (z+1)\cos\phi$$

$$= 3(z+1)\cos\phi \quad \mu C/m^3.$$



b)

$Q = \int \rho_v \, dv.$

Differential volume $(dv) = \rho\, d\rho\, d\phi\, dz$

$$\therefore Q = \int 3(z+1)\cos\phi\, \rho\, d\rho\, d\phi\, dz$$

$$= 3\int_0^4 (z+1)\,dz \int_0^{\pi/2} \cos\phi\, d\phi \int_0^2 \rho\, d\rho$$

$$= 3.\left|\frac{z^2}{2}+z\right|_0^4 \cdot \left|\sin\phi\right|_0^{\pi/2} \left|\frac{\rho^2}{2}\right|_0^2$$

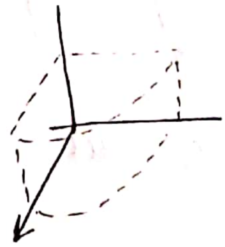
$$= 3.(8+4).1.2 = 72\ \mu C.$$

4.30

c) Confirm Gauss Law

$$\oint \bar{D}.d\bar{s} = \int_{top} \bar{D}.d\bar{s} + \int_{bottom} \bar{D}.d\bar{s} + \int_{left} \bar{D}.d\bar{s}$$

$$+ \int_{right} \bar{D}.d\bar{s} + \int_{front} \bar{D}.d\bar{s}$$



$d\bar{s}$ (for top) $= \rho\, d\rho\, d\phi\, \hat{a_z}$.

$$\int_{top} \bar{D}.d\bar{s} = \iint \rho^2 \cos\phi . \rho d\rho d\phi$$

$$= \int_0^2 \int_{\phi=0}^{\phi=\pi/2} \rho^3 \cos\phi \, d\rho\, d\phi = \int_0^2 \rho^3 d\rho \int_0^{\pi/2} \cos\phi\, d\phi$$

$$= \frac{\rho^4}{4}\Big|_0^2 \sin\phi \Big|_0^{\pi/2}.$$

$$= 4.1 = 4.\ \mu C$$

$d\bar{s}$ (for bottom) $= \rho d\rho d\phi (-\hat{a_z})$

$$\int_{bottom} \bar{D}.d\bar{s} = \int_0^2 \int_0^{\pi/2} -\rho^3 \cos\phi \, d\rho\, d\phi$$

$$= \int_0^2 \rho^3 d\rho \int_0^{\pi/2} -\cos\phi\, d\phi$$

$$= \frac{\rho^4}{4}\Big|_0^2 \left|-\sin\phi\right|_0^{\pi/2} = -4.\ \mu C$$

$d\bar{s}$ (for left) $= d\rho dz (-\hat{a_\phi})$

$$\int_{left} \bar{D}.d\bar{s} = \iint -\rho(z+1)\sin\phi (d\rho dz)$$

$$\int_{left} \bar{D}.d\bar{s} = \int_0^2 \rho\, d\rho \int_0^4 (z+1)\sin\phi\, dz. \qquad \phi = 0^\circ \text{ (for this surface)}$$

$\therefore \sin\phi = 0$

$$= 0 \quad \mu C$$

$$\int_{right} \bar{D}.d\bar{s} = \iint -\rho(z+1)\sin\phi\, d\rho\, dz$$

$d\bar{s}$ (for right) $= d\rho\, dz\, \hat{a}_\phi$

$$= \int_0^2 -\rho\, d\rho \int_0^4 (z+1)\sin\phi\, dz.$$

$\phi = \pi/2$ (for this surface)

$$= \left|\frac{-\rho^2}{2}\right|_0^2 \left|\frac{z^2}{2}+z\right|_0^4$$

$$= -2\,(8+4) = -24 \ \mu C$$

$d\bar{s}$ (front) $= \rho\, d\phi\, dz\, \hat{a}_\rho.$

$$\int_{front} \bar{D}.d\bar{s} = \iint 2\rho(z+1)\cos\phi\,.\,\rho\, d\phi\, dz$$

$$= 2\rho^2 \int_{\phi=0}^{\phi=\pi/2} \cos\phi\, d\phi \int_0^4 (z+1)\, dz$$

$\rho = 2$ (for this surface).

$$= 8\left|\sin\phi\right|_{\phi=0}^{\phi=\pi/2} \left|\frac{z^2}{2}+z\right|_0^4$$

$$= 8\,(8+4) = 96 \ \mu C.$$

$$\oint \bar{D}.d\bar{s} = 4 - 4 + 0 - 24 + 96 = 72\,\mu C \text{ (Proved).}$$

3) Electric field intensity in free space is given by

$$\bar{E} = 2xyz\,\hat{a}_x + x^2z\,\hat{a}_y + x^2y\,\hat{a}_z \ \text{V/m}.$$

Position $P_1 = (2,1,-1)$

Position $P_2 = (5,1,2)$

Work done ~~necessary~~ to move $(Q = 2\mu C)$ charge from P_1 to P_2

$$W = -Q\int_{P_1}^{P_2} \bar{E}\cdot d\bar{l} \qquad d\bar{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

(since there is no change in y).

$$= -2\times10^{-6}\int_{(2,1,-1)}^{(5,1,2)} \left(2xyz\,dx + \cancel{x^2z\,dy}^{\,0} + x^2y\,dz\right)$$

$$\cancel{W_1 = -2\times10^{-6}\int_{(2,1,-1)}^{(5,1,2)} 2xyz\,dx = -2\times10^{-6}\,yz}$$

$$W_p = -2\times10^{-6}\int_{(2,1,-1)}^{(5,1,-1)} 2xyz\,dx + (-2\times10^{-6})\int_{(5,1,-1)}^{(5,1,2)} x^2y\,dz.$$

$$= -2\times10^{-6}\times1\times(-1)\left|\frac{x^2}{2}\right|_2^5 - 2\times10^{-6}\times5^2\times1\times\left|z\right|_{-1}^{2}$$

$$= +2\times10^{-6}\left(\frac{25-4}{2}\right) - 2\times10^{-6}\times25\times(2+1)$$

$$= 21\times10^{-6} - 150\times10^{-6} = -129\times10^{-6}\ \mu J.$$

b) Show $\bar{E}$ is conservative.

$$\nabla \times \bar{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a_x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a_y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a_z}$$

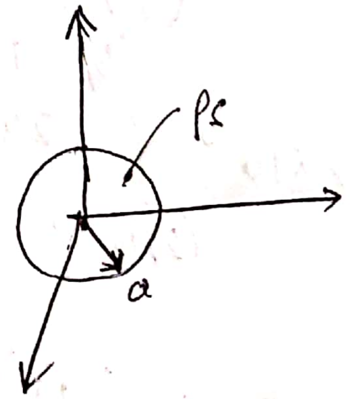
$$= \left[\frac{\partial}{\partial y}(x^2y) - \frac{\partial}{\partial z}(x^2z)\right]\hat{a_x} + \left[\frac{\partial}{\partial z}(2xyz) - \frac{\partial}{\partial x}(x^2y)\right]\hat{a_y}$$

$$+ \left[\frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(2xyz)\right]\hat{a_z}$$

$$= (x^2 - x^2)\hat{a_x} + (2xy - 2xy)\hat{a_y} + (2xz - 2xz)\hat{a_z}$$

$= 0.$ (Proved).

4.> A circular disk of radius 'a' carries a charge $\rho_s = \frac{1}{\rho}\ C/m^2$. Calculate potential V at $P(0,0,h)$



$\bar{r} = (0,0,h)\ m$

$\bar{r}' = \rho'\hat{a_\rho}\ m.$

$ds' = \rho' d\rho' d\phi'\ m^2$

$|\bar{r} - \bar{r}'| = \sqrt{\rho'^2 + h^2}$

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_s}{|\bar{r} - \bar{r}'|}ds' = \frac{\rho_s}{4\pi\epsilon_0}\int \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + h^2}}$$

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{(\lambda/\rho')\rho' d\rho' d\phi'}{\sqrt{\rho'^2+h^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^a \frac{d\rho'}{\sqrt{\rho'^2+h^2}}$$

Let, $I = \int_0^a \frac{d\rho'}{\sqrt{\rho'^2+h^2}}$

$u = \rho'/h$.

$du = \frac{1}{h}d\rho$

$$= \int \frac{1/h \, d\rho'}{\sqrt{(\rho'/h)^2+1}}$$

Put, $u = \tan v$

$du = \sec^2 v \, dv$

$$= \int \frac{du}{\sqrt{u^2+1}}$$

$$= \int \frac{\sec^2 v \, dv}{\sec v}$$

$$= \int \sec v \, dv$$

$$= \int \frac{\sec v(\tan v + \sec v)}{\tan v + \sec v} dv$$

Let, $\tan v + \sec v = w$.

$\therefore dw = (\sec v \tan v + \sec^2 v)dv$.

$$= \int \frac{1}{w} dw = \ln(w)$$

$$= \ln(\tan v + \sec v)$$

$$= \ln(u + \sqrt{u^2+1})$$

$u = \rho'/h$

$$I = \ln\left[\sqrt{(\rho'/h)^2+1} + \rho'/h\right]$$

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left|\phi'\right|_0^{2\pi}\left[\ln\sqrt{(\rho'/h)^2+1} + \rho'/h\right]_0^a .$$

$$= \frac{2\pi}{4\pi\epsilon_0}\cdot\ln\left\{\sqrt{(a/h)^2+1} + a/h\right\}$$

$$= \frac{1}{2\epsilon_0}\ln\left(\frac{\sqrt{a^2+h^2}+a}{h}\right) \text{ V.}$$

5.) a) Potential $V = 10\rho^2\sin\phi + 6\rho z$ V

$$\bar{E} = -\nabla.\vec{V}$$

$$= -\frac{\partial V}{\partial\rho}\hat{a_\rho} - \frac{1}{\rho}\frac{dV}{\partial\phi}\hat{a_\phi} - \frac{\partial V}{\partial z}\hat{a_z}.$$

$$= -(20\rho\sin\phi + 6z)\hat{a_\rho} - \frac{1}{\rho}(10\rho^2\cos\phi)\hat{a_\phi} - 6\rho\hat{a_z}$$

$$= -(20\rho\sin\phi + 6z)\hat{a_\rho} - 10\rho\cos\phi\,\hat{a_\phi} - 6\rho\,\hat{a_z} \text{ V/m.}$$

b) $V = 5r^2 \cos\theta \sin\phi$ V.

$\bar{E} = -\nabla . V$

$= -\left[\frac{\partial V}{\partial r}\hat{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a_\theta} + \frac{1}{r\sin\theta}\frac{dV}{\partial \phi}\hat{a_\phi}\right]$

$= -\left[\frac{\partial}{\partial r}(5r^2\cos\theta\sin\phi)\hat{a_r} + \frac{1}{r\partial\theta}(5r^2\cos\theta\sin\phi)\hat{a_\theta} + \frac{1}{r\sin\theta}\frac{d}{\partial\phi}(5r^2\cos\theta\sin\phi)\hat{a_\phi}\right]$

$= -\left[10r\cos\theta\sin\phi\,\hat{a_r} + \cancel{} - 5r\sin\theta\sin\phi\,\hat{a_\theta} + 5r\cot\theta\cos\phi\,\hat{a_\phi}\right]$

$= -10r\cos\theta\sin\phi\,\hat{a_r} + 5r\sin\theta\sin\phi\,\hat{a_\theta} - 5r\cot\theta\cos\phi\,\hat{a_\phi}$ V/m.