Homework #1 EE 331

Solutions

2(a) Taylor's series

Let, a be an initial point along the curve $f(x)$.

Let, h be a small increment to a.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f''''(a) + \ldots$$

Put, $a=0$,

h being a dummy variable, replace h by x.

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \ldots$$

So, if derivatives, first derivatives, second derivatives of function $f(x)$ at $x=0$ is known, then $f(x)$ can be represented as a sum of infinite number of polynomials.

$f(x) = e^{-x}$

$f(0) = e^0 = 1$.

$f'(x) = -e^{-x} \Rightarrow f'(0) = -e^{-x}\big|_{x=0} = -1$.

$f''(0) = \frac{\partial}{\partial x}(f'(x))\big|_{x=0} = e^{-x}\big|_{x=0} = 1$.

$f'''(0) = -1$

$f''''(0) = 1$.

$\therefore$ Ignoring the higher order terms,

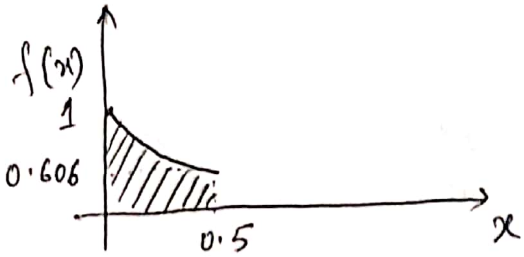
$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$$

$$\Rightarrow e^{-x} \approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \quad \text{(Ans)}.$$

2b)
$$I = \int_0^{0.5} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^{0.5}$$
$$= -1\left(e^{-0.5} - 1\right)$$
$$= 0.3934. \text{ (Ans).}$$

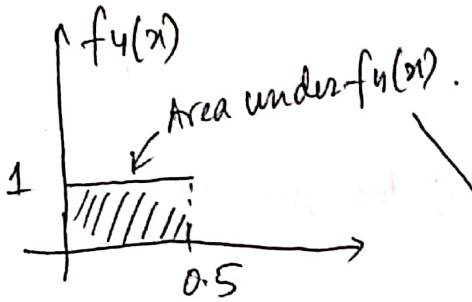
2c)
$$I_4 = \int_0^{0.5} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right) dx.$$
$$= \int_0^{0.5} \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) dx$$
$$= \left[x\right]_0^{0.5} - \left[\frac{x^2}{2}\right]_0^{0.5} + \frac{1}{2}\left[\frac{x^3}{3}\right]_0^{0.5} - \frac{1}{6}\left[\frac{x^4}{4}\right]_0^{0.5}$$
$$= 0.5 - \frac{0.25}{2} + \frac{1}{6}(0.5)^3 - \frac{1}{24}(0.5)^4$$
$$= 0.393229.$$

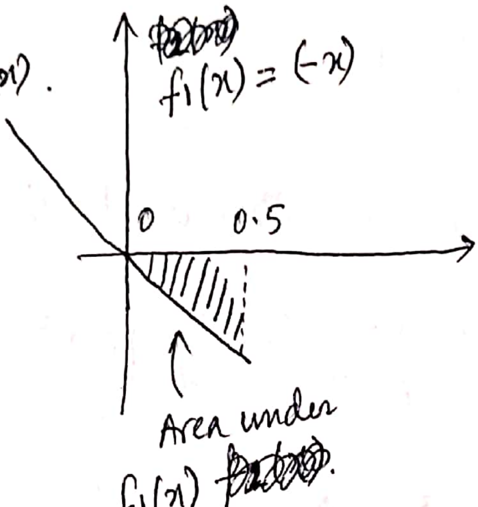
2d) ~~Taylor's series provides~~
Any function $f(x)$ can be replaced by a series of polynomials using Taylor's series.
This is proved using the integration problem. 2(b) does an integration of the curve $f(x) = e^{-x}$. This means we have to find out the area under the curve e^{-x} from $x = 0$ to $x = 0.5$.

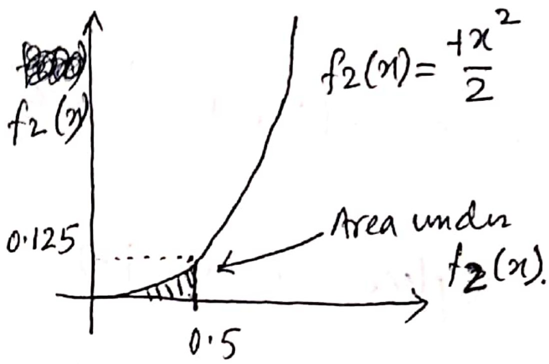


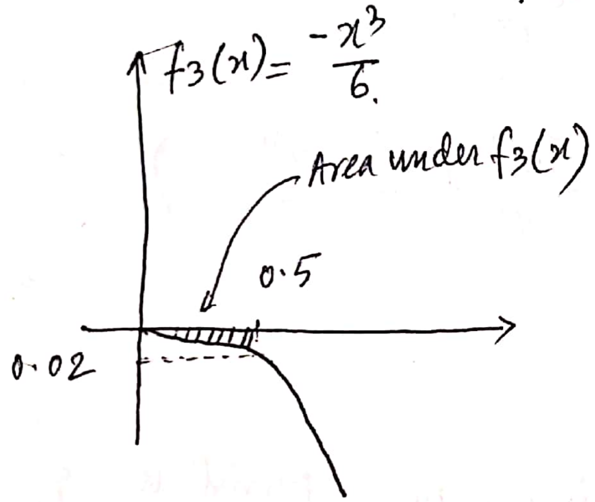
The shaded region is the area under the curve e^{-x} from $x=0$ to $x=0.5$.

n 2(c), we have to find out the area under the curves of the polynomials $f_1(x) = -x$, $f_2(x) = \frac{x^2}{2}$, $f_3(x) = \frac{-x^3}{3!}$ and $f_4(x) = 1$.









If the area under e^{-x} and the sum of areas under $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ are approximately equal, we can say that the Taylor's series gives a good approximation of a function as sum of several polynomials.

3(a). From Taylor's series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\cdots \rightarrow (1)$$

Let, $f(x) = e^x$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \cdots$$

$$\Rightarrow e^x = 1 + x\left[e^x\right]_{x=0} + \frac{x^2}{2!}\left[e^x\right]_{x=0} + \frac{x^3}{3!}\left[e^x\right]_{x=0} + \frac{x^4}{4!}\left[e^x\right]_{x=0} + \cdots$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Replace x by (ix) where $i =$ unit in the imaginary axis.

$$\therefore e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)$$

$$= \cos x + i \sin x.$$

$$\left.\begin{aligned} \cos x &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right) \\ \sin x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right) \end{aligned}\right\}$$

This can be proved using Taylor's series

Replace ix by $(-ix)$

$$e^{-ix} = \cos x - i \sin x.$$

$\therefore$ $\cancel{e^{ix} + (e^{-ix})}$

$$e^{ix} - e^{-ix} = \cos x + i\sin x - \cos x + i \sin x = 2i \sin x.$$

$$\therefore \sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{(Proved).}$$

3b) $Z = 4 - j3.$

Any complex quantity has an amplitude and argument or angle.

$$Z = |Z| \angle \theta = |Z|(\cos\theta + j\sin\theta)$$

$\therefore$ $|Z| \cos\theta = 4$

$|Z| \sin\theta = -3.$

$\therefore$ $\tan\theta = \frac{-3}{4} \Rightarrow \theta = \tan^{-1}(-3/4) = -36.86°.$

$\therefore$ $\cos\theta = 4/5$

$\sin\theta = -3/5.$

$$\therefore |Z| = \frac{4}{\cos\theta} = \frac{4}{4/5} = 5.$$

$\therefore$ $Z = 5\angle -36.86°$ (Ans).

3c)

$$Z = \frac{1-j2}{2-j} = \frac{(1-j2)(2+j)}{(2+j)(2-j)}$$

$$= \frac{2-j4+j+2}{4+1}$$

$$= \frac{4-j3}{5}$$

4(a). Any time-harmonic field can be expressed in these two forms.

$$\tilde{A}(x,y,z,t) = Re\left[\underbrace{A(x,y,z)}_{\text{Phasor form.}} e^{j\omega t}\right]$$

$\downarrow$

Time-domain form

$V(z,t) = 5\cos(5t - kz + 0.2\pi)$ $\leftarrow$ Time Domain form

$$V(z,t) = Re\left[5\, e^{j(5t - kz + 0.2\pi)}\right]$$

$$= Re\left[5 e^{+j(-kz+0.2\pi)} e^{j5t}\right].$$

$\therefore$ Phasor form, $V(z) = 5e^{-j(kz - 0.2\pi)}$

4b) $V_s(z) = 10\, e^{-(2-j3)z + j0.3\pi}$

$$V(z,t) = Re\left[10 e^{-(2-j3)z + j0.3\pi} . e^{j\omega t}\right].$$

$$= 10 e^{-2z} Re\left[e^{j(\omega t + 3z + 0.3\pi)}\right] = 10 e^{-2z} \cos(\omega t + 3z + 0.3\pi)$$