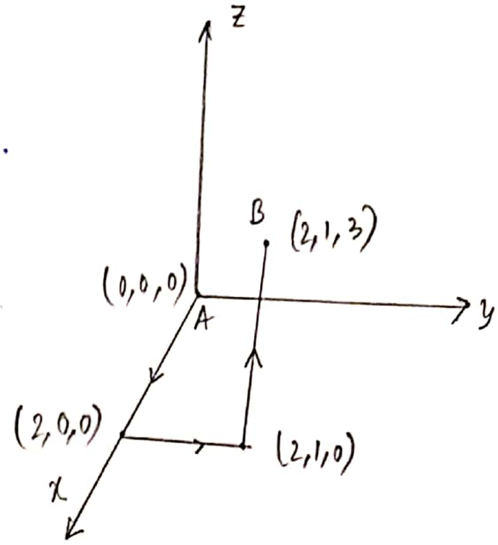
Homework # 8

1.)

$$\bar{A} = 2xy\,\hat{a}_x + x^2 z\,\hat{a}_y + x^2 y\,\hat{a}_z$$

$$\int_A^B \bar{A}\cdot d\bar{l} \qquad dl = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z.$$



$$= \int_{(0,0,0)}^{(2,1,3)} 2xy\,dx + x^2 z\,dy + x^2 y\,dz$$

$$= \int_{(0,0,0)}^{(2,0,0)} 2xy\,dx + \int_{(2,0,0)}^{(2,1,0)} x^2 z\,dy + \int_{(2,1,0)}^{(2,1,3)} x^2 y\,dz$$

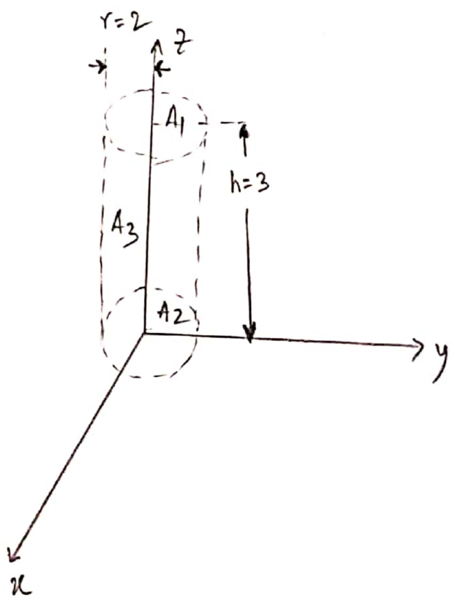
$$= 8y\cdot\frac{x^2}{2}\Bigg|_{x=0,\,y=0,\,z=0}^{x=2,\,y=0,\,z=0} + x^2yz\Bigg|_{x=2,\,y=0,\,z=0}^{x=2,\,y=1,\,z=0} + x^2yz\Bigg|_{x=2,\,y=1,\,z=0}^{x=2,\,y=1,\,z=3}$$

$$= \cancel{4}\ 0 + 0 + \cancel{4\times3} = 2^2.1.3 = 12.$$

2.)

$$\bar{B} = \rho^2\hat{a}_\rho + 2z\,\hat{a}_\phi + \cos\phi\,\hat{a}_z$$

upper surface → A_1
lower surface → A_2
Side → A_3.



For surface A_1

$$\oint_S \bar{B}.d\bar{s} = \int_{A_1} \bar{B}.d\bar{s} + \int_{A_2} \bar{B}.d\bar{s} + \int_{A_3} \bar{B}.d\bar{s}.$$

For surfaces A_1 & A_2 & A_3

$$dS_{A_1} = \rho\, d\rho\, d\phi\, \hat{a_z}$$
$$dS_{A_2} = -\rho\, d\rho\, d\phi\, \hat{a_z}$$
$$dS_{A_3} = \rho\, d\phi\, dz\, \hat{a_\rho}.$$

$$\int_{A_1} \cancel{\oint} \bar{B}.d\bar{s} = \cancel{\oint} \int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=2} \rho\, d\rho\, d\phi . \cos\phi = \int_0^2 \rho\, d\rho \int_0^{2\pi} \cos\phi\, d\phi = 0$$

$$\cancel{\oint} \int_{A_2} \bar{B}.d\bar{s} = \int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=2} -\rho\, d\rho\, d\phi \cos\phi = -\int_0^2 \rho\, d\rho \int_0^{2\pi} \cos\phi\, d\phi = 0.$$

$$\int_{A_3} \bar{B}.d\bar{s} = \iint \bar{B}.\, \rho\, d\phi\, dz\, \hat{a_\rho}.$$

$$= \int_{\phi=0}^{2\pi}\int_{z=0}^{z=3} \rho^3 d\phi\, dz = \int_0^3 \rho^3 dz \int_0^{2\pi} d\phi = \rho^3 \int_0^3 dz \int_0^{2\pi} d\phi$$

$$= 2^3 . 3 . 2\pi$$

$$= 48\pi.$$

$$\boxed{\therefore \oint_S \bar{B}.d\bar{s} = 48\pi}$$

3.)

$$\rho_v = 0 \qquad 0 \le r < 2$$
$$= \frac{2}{r^2}\ nC/m^3 \qquad 2 \le r \le 4$$
$$= 0 \qquad r > 4.$$

$$Q = \int \rho_v dv = 2\times10^{-9} \cdot \frac{1}{r^2} \iiint r^2 \sin\theta\, dr\, d\theta\, d\phi.$$
$$= (2\times10^{-9}) \int_2^4 dr \int_0^{\pi} \sin\theta \int_0^{2\pi} d\phi$$
$$= (2\times10^{-9})\, r\Big|_2^4 \cdot \cos\theta\Big|_{\pi}^{0}\ \phi\Big|_0^{2\pi}.$$
$$= (2\times10^{-9}) \times 2\times2\times2\pi = 16\pi\ nC.$$

4.) a)

$$T = e^{x+2y}\cosh z$$
$$\nabla . T = \frac{\partial T}{\partial x}\hat{a_x} + \frac{\partial T}{\partial y}\hat{a_y} + \frac{\partial T}{\partial z}\hat{a_z}.$$
$$= \frac{\partial}{\partial x}\left(e^{x+2y}\cosh z\right)\hat{a_x} + \frac{\partial}{\partial y}\left(e^{x+2y}\cosh z\right)\hat{a_y}$$
$$+ \frac{\partial}{\partial z}\left(e^{x+2y}\cosh z\right)\hat{a_z}$$
$$= e^{x+2y}\cosh z\ \hat{a_x} + 2e^{x+2y}\cosh z\ \hat{a_y}$$
$$+ e^{x+2y}\sinh z\ \hat{a_z}$$

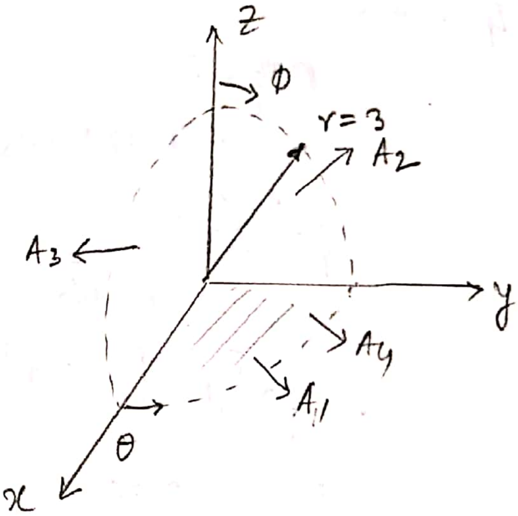
b) $U = \frac{3z}{\rho}\cos\phi$

$$\nabla U = \frac{\partial}{\partial\rho} U \hat{a}_\rho + \frac{1}{\rho}\frac{d}{\partial\phi} U \hat{a}_\phi + \frac{\partial}{\partial z} U \hat{a}_z .$$

$$= 3z\cos\phi \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right)\hat{a}_\rho + \frac{1}{\rho^2} 3z \frac{\partial}{\partial\phi}(\cos\phi)\,\hat{a}_\phi + \frac{3\cos\phi}{\rho}\frac{d}{\partial z}(z)\,\hat{a}_z$$

$$= -\frac{3z}{\rho^2}\cos\phi\,\hat{a}_\rho - \frac{3z}{\rho^2}\sin\phi\,\hat{a}_\phi + \frac{3\cos\phi}{\rho}\hat{a}_z .$$

5) Verify Divergence Theorem for $\bar{A} = r^2\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta$



Divergence Theorem

$$\oint_S \bar{A}\cdot d\bar{s} = \int_V \nabla\cdot\bar{A}\,dv$$

$$\oint_S = \int_{A_1} + \int_{A_2} + \int_{A_3} + \int_{A_4}$$ A_4 = bottom.

$$\int_{A_1}\bar{A}\cdot d\bar{s} = \cancel{\int_{A_1}\bar{A}\cdot \rho\, d\phi\, dz\, \hat{a}_\rho} = \int_{A_1}\bar{A}\cdot r^2\sin\theta\, d\theta\, d\phi\,\hat{a}_r$$

$$= \iint r^4\sin\theta\, d\theta\, d\phi$$

$$= r^4\int_0^{\pi/2}\sin\theta\, d\theta \int_0^{\pi/2} d\phi$$

$$\int_{A_1} \bar{A}.ds = r^4 \int_0^{\pi/2} \sin\theta d\theta \int_0^{\pi/2} d\phi \qquad r = 3$$

$$= 3^4 . \left| -\cos\theta \right|_0^{\pi/2} (\pi/2 - 0)$$

$$= 3^4 . \frac{\pi}{2} \left| \cos\theta \right|_{\pi/2}^{0}$$

$$= \frac{81.\pi}{2}$$

$$\int_{A_4 = \text{bottom}} \bar{A}.d\bar{s} = \int_{A_4} \bar{A}.(r\sin\theta dr d\phi \hat{a_\theta})$$

$$= \int_{\phi=0}^{\phi=\pi/2} \int_{r=0}^{r=3} r\sin\theta\cos\phi . r\sin\theta dr d\phi$$

$$= \int_0^3 r^2 dr \int_{\phi=0}^{\phi=\pi/2} \sin^2\theta \cos\phi d\phi \qquad \theta = \pi/2$$

$$= \sin^2(\pi/2) . \frac{r^3}{3}\Big|_0^3 \sin\phi\Big|_0^{\pi/2}$$

$$= 1 . \frac{27}{3} . 1 = 9.$$

$$\int_{A_2} \bar{A}.d\bar{s} = \int_{A_2} \bar{A}. r dr d\theta \hat{a_\phi} = 0.$$

$$\int_{A_3} \bar{A}.d\bar{s} = \int_{A_3} \bar{A}. r dr d\theta \hat{a_\phi} = 0.$$

$$\oint_S \bar{A}.d\bar{s} = \int_{A_1} \bar{A}.d\bar{s} + \int_{A_4} \bar{A}.d\bar{s}$$

$$= \frac{81\pi}{2} + 9.$$

Divergence of $\bar{A}$

$$\bar{\nabla}.\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}.0$$

$$= 4r + \frac{1}{r\sin\theta}.\cos\phi.r.2\sin\theta\cos\theta$$

$$= 4r + 2\cos\theta\cos\phi.$$

$$\int_V \bar{\nabla}.\bar{A}\,dv = \int_V (4r + 2\cos\theta\cos\phi)(r^2\sin\theta\,dr\,d\theta\,d\phi).$$

$$= 4\int_0^3 r^3 dr \int_0^{\pi/2}\sin\theta\,d\theta \int_0^{\pi/2} d\phi + 2\int_0^3 r^2 dr \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \int_0^{\pi/2}\cos\phi\,d\phi$$

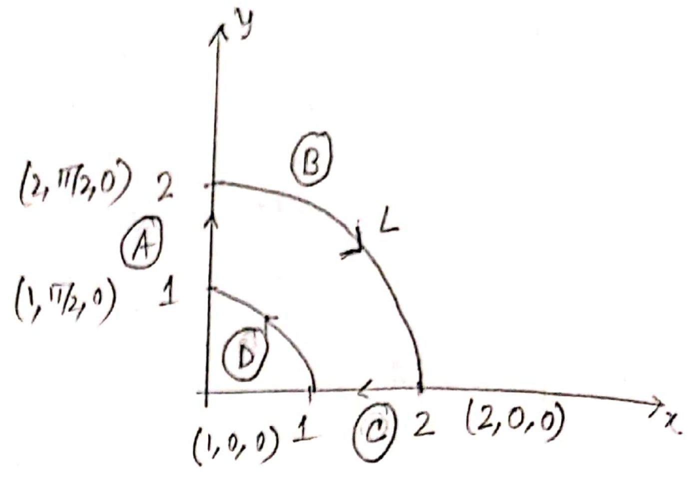
$$= r^4\Big|_0^3 \cos\theta\Big|_{\pi/2}^0 \phi\Big|_0^{\pi/2} + \frac{2}{3}r^3\Big|_0^3 \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} \sin\phi\Big|_0^{\pi/2}$$

$$= 81.\pi/2 + 18.\frac{1}{2}.1 = \frac{81\pi}{2} + 9.$$

$$\therefore \oint_S \bar{A}\cdot d\bar{s} = \int_V \nabla\cdot\bar{A}\, dv.$$

6.) $\bar{A} = \rho\sin\phi\,\hat{a}_\rho + \rho^2\hat{a}_\phi$.

a) The paths Ⓐ, Ⓑ, Ⓒ & Ⓓ are shown.



$$\oint_L \bar{A}\cdot dL = \int_{Ⓐ} \bar{A}\cdot d\bar{l}_{Ⓐ} + \int_{Ⓑ} \bar{A}\cdot d\bar{l}_{Ⓑ} + \int_{Ⓒ} \bar{A}\cdot d\bar{l}_{Ⓒ} + \int_{Ⓓ} \bar{A}\cdot d\bar{l}_{Ⓓ}$$

The four points are represented in cylindrical coordinates.

Along Ⓐ, only radius (ρ) changes, $d\rho$ = non-zero, $\phi = \pi/2$, $z = 0$.

$$\int_{Ⓐ} \bar{A}\cdot d\bar{l}_{Ⓐ} = \int_{\rho=1}^{\rho=2} \rho\sin\phi\, d\rho \qquad (\phi = \pi/2).$$

$$= \sin\pi/2 \cdot \frac{\rho^2}{2}\Big|_1^2 = \frac{1}{2}\cdot(4-1) = \frac{3}{2}.$$

Along Ⓑ, radius (ρ) remains same as well as z. But, ϕ changes from $\pi/2$ to 0.

$$\int_{Ⓑ} \bar{A}\cdot d\bar{l}_{Ⓑ} = \int_{\pi/2}^{0} \rho\, d\phi \cdot \rho^2 \qquad (\rho = 2)$$

$$= \rho^3\,\phi\Big|_{\pi/2}^{0} = 8(-\pi/2) = -4\pi.$$

Along ©, both ϕ and z remain unchanged while ρ ~~is~~ changes from 2 to 1.

$$\int_{©} \bar{A}\cdot d\bar{l}_{©} = \int_{\rho=2}^{\rho=1} \rho \sin\phi \cdot d\rho \qquad (\phi = 0).$$

$$= 0.$$

$$\int_{(D)} \bar{A}\cdot d\bar{l}_{(D)} = \int_{\phi=0}^{\phi=\pi/2} \rho^2 \cdot \rho\, d\phi \qquad (\rho = 1)$$

$$= 1 \cdot \phi \Big|_0^{\pi/2} = \pi/2$$

$$\therefore \oint \bar{A}\cdot d\bar{l} = \frac{3}{2} - 4\pi + 0 + \pi/2 = -9.4955.$$

b) $$\nabla \times \bar{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{a_\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{a_\phi}$$

$$+ \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{a_z}.$$

$$= 0 + 0 + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^3) - \rho\cos\phi\right]\hat{a_z}.$$

$$= (3\rho - \cos\phi)\,\hat{a_z}.$$

$$\int(\nabla\times\bar{A})\cdot d\bar{s} = \iint(3\rho-\cos\phi)\hat{a_z}\cdot(-\rho\, d\rho\, d\phi)\hat{a_z}$$

$$= \int_{\phi=0}^{\pi/2}\int_{\rho=1}^{\rho=2} -3\rho^2 d\rho\, d\phi + \int_{\phi=0}^{\phi=\pi/2}\int_{\rho=1}^{\rho=2} \rho\cos\phi\, d\rho\, d\phi.$$

$$= -\frac{3}{3}\rho^3\Big|_1^2 \phi\Big|_0^{\pi/2} + \frac{\rho^2}{2}\Big|_1^2 \sin\phi\Big|_0^{\pi/2}$$

$$= \cancel{-18\cdot\pi/2 + \frac{...}{2}}$$

$$= -(2^3-1^3)\cdot\pi/2 + \frac{1}{2}(4-1)\cdot 1$$

$$= -7\cdot\pi/2 + \frac{3}{2} = -9.4955.$$

7.) Prob. 3.56(a).

$V = x^3y^2e^{xz}$ find Laplacian at $(1,-1,1)$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(3x^2y^2e^{xz}\right) + \frac{\partial}{\partial x}\left(x^3y^2ze^{xz}\right) + \frac{\partial}{\partial y}\left(2x^3y\, e^{xz}\right)$$

$$+ \frac{\partial}{\partial z}\left(x^3y^2xe^{xz}\right)$$

$$\nabla^2 U = 6xy^2e^{xz} + 3x^2y^2 \cdot z e^{xz} + 3x^2y^2ze^{xz} + x^3y^2z^2e^{xz}$$
$$+ 2x^3e^{xz} + x^3y^2x^2e^{xz}$$

$\Rightarrow$ $\cancel{6xy^2e^{xz} + 3x^2y^2ze^{xz}}$

$$= e^{xz}\left[6xy^2 + 3x^2y^2z + 3x^2y^2z + x^3y^2z^2 + 2x^3 + x^5y^2\right].$$

@ $P(1,-1,1) = \nabla^2 U = e^1[6+6+1+2+1] = 43.4925.$