EE331 Homework 9

① Problem 4.4.

$m = 2kg$, $Q = -4\,mC = -4\times10^{-3}$ Coulomb.

Force of gravity $(\bar{F}_G) = -mg\,\hat{a}_z$ [$\hat{a}_z$ is perpendicular "upward" from earth's surface.

Force due to Electric field intensity $(\bar{E}) \Rightarrow F_E = Q\bar{E}$

$$\bar{F}_G + \bar{F}_E = 0$$

$g = 9.81\,m/s^2$

$$\Rightarrow -mg\,\hat{a}_z + Q\bar{E} = 0.$$

$$\Rightarrow \bar{E} = \frac{mg}{Q}\hat{a}_z = \frac{2\times 9.81}{-4\times10^{-3}} = -4.9050\,\hat{a}_z\ kV/m.$$

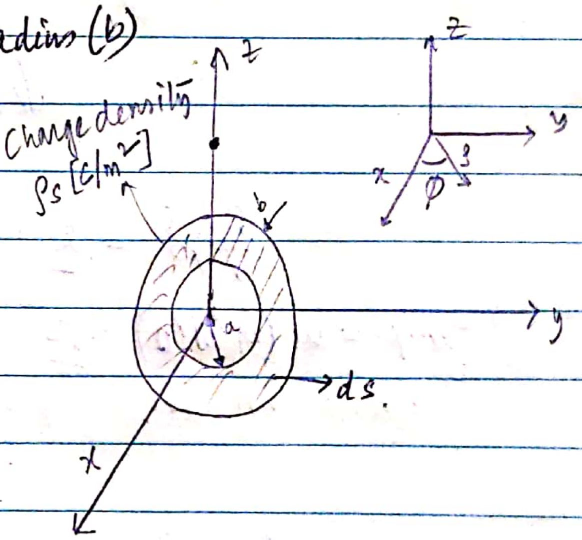
Hence the electric field intensity $(\bar{E})$ is directed "downward" perpendicular to the earth's surface. This is due to presence of negative electric charge.

② Prob 4.13

Annular disc of inner radius (a) and outer radius (b) is placed on the x-y plane.

The surface charge density in the annular region is ρ_s [C/m²]

The differential surface $d\bar{s} = \rho\, d\phi\, dz\, \hat{a}_\rho$.



Electric field intensity at (0, 0, h) due to annular surface is given by $\bar{E}$.

$$\bar{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_s(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\, ds \qquad \bar{r} = (0,0,h)\, m = h\hat{a_z}, \quad \bar{r}' = \rho'\hat{a_\rho}$$

$$= \frac{\rho_s}{4\pi\epsilon_0}\int \frac{h\hat{a_z} - \rho'\hat{a_\rho}}{|h\hat{a_z} - \rho'\hat{a_\rho}|^3} \cdot \rho' d\rho' d\phi$$

Owing to the symmetry of charge distribution, $\rho\hat{a_\rho}$ component can be cancelled to 0. (Check 4.3.B, Pg-123)

$$\therefore \bar{E} = \frac{\rho_s}{4\pi\epsilon_0}\int \frac{h\hat{a_z}}{(h^2+\rho'^2)^{3/2}}\, \rho d\rho' d\phi.$$

$$= \frac{\rho_s \cdot h}{4\pi\epsilon_0}\int_a^b \frac{\rho' d\rho'}{(h^2+\rho'^2)^{3/2}} \int_0^{2\pi} d\phi \;\; \hat{a_z}.$$

Let, $h^2 + \rho'^2 = v^2$

$2\rho' d\rho' = 2v dv$

$\rho' = a, \; v = (h^2+a^2)^{1/2}$

$\rho' = b, \; v = (h^2+b^2)^{1/2}$

$$= \frac{\rho_s h}{4\pi\epsilon_0}\int_{(h^2+a^2)^{1/2}}^{(h^2+b^2)^{1/2}} \frac{v dv}{v^3} \int_0^{2\pi} d\phi \;\; \hat{a_z}.$$

$$= \frac{\rho_s \cdot h}{4\pi\epsilon_0}\int_{\sqrt{h^2+a^2}}^{\sqrt{h^2+b^2}} v^{-2} dv \int_0^{2\pi} d\phi \cdot \hat{a_z}$$

$$= \frac{\rho_s . h}{4\pi\epsilon_0} \left[-\frac{1}{v}\right]_{\sqrt{h^2+a^2}}^{\sqrt{h^2+b^2}} \left[\phi\right]_0^{2\pi} \hat{a}_z$$

$$= \frac{\rho_s h}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{h^2+a^2}} - \frac{1}{\sqrt{h^2+b^2}}\right] 2\pi \hat{a}_z$$

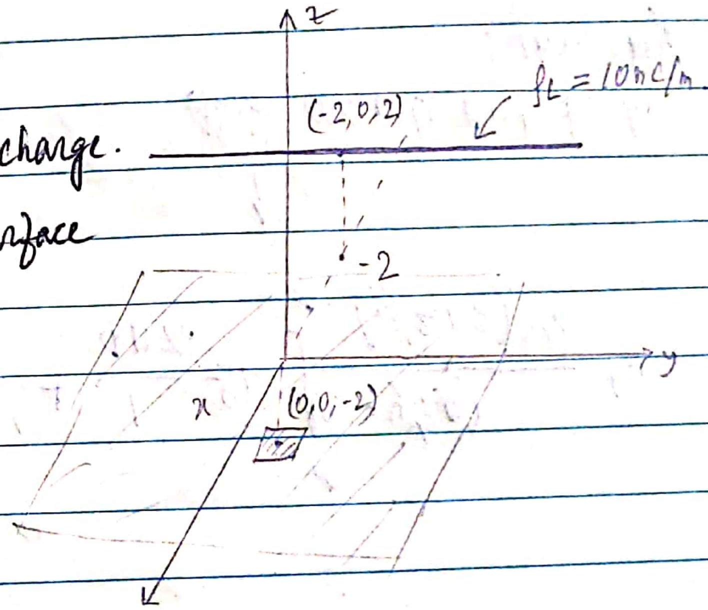
$$= \frac{\rho_s . h}{2\epsilon_0} \left[\frac{1}{\sqrt{h^2+a^2}} - \frac{1}{\sqrt{h^2+b^2}}\right] \hat{a}_z \ V/m.$$

(3) Line $(-2,y,2)$ carries a charge of $10nC/m$ while plane $z=-2$ carries a charge of $4nC/m^2$. Find electric field at the origin due to these charges.

$\bar{r} = (0,0,0)\,m \rightarrow$ Position vector at origin

$\bar{r}_L = (-2,y',2)\,m \rightarrow$ Position vector of line charge.

$\bar{r}_s = (x',y',-2)\,m \rightarrow$ Position vector of surface charge.



$\bar{r} - \bar{r}_L = (2,-y',-2)\,m$

$\bar{r} - \bar{r}_s = (-x',-y',2)\,m$

$|\bar{r} - \bar{r}_L|^3 = \left[y'^2+8\right]^{3/2} m^3$

$|\bar{r} - \bar{r}_s|^3 = \left[x'^2+y'^2+4\right]^{3/2} m^3.$

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Electric field due to line charge at origin of density $\rho_L = \bar{E}_L$

$$\bar{E}_L = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_L(\bar{r}-\bar{r}_L')\,dL'}{|\bar{r}-\bar{r}_L'|^3}$$

$dL' = dy'$ (Since line charge is along y-axis)

$$= \frac{1}{4\pi\epsilon_0}\rho_L \int_{-\infty}^{\infty} \frac{(2,-y',-2)}{(y'^2+8)^{3/2}}\,dy'.$$

$$= \frac{1}{4\pi\epsilon_0}\rho_L \int_{-\infty}^{\infty} \frac{(2,0,-2)}{(y'^2+8)^{3/2}}\,dy' + \frac{1}{4\pi\epsilon_0}\rho_L \int_{-\infty}^{\infty} \frac{(0,-y',0)}{(y'^2+8)^{3/2}}\,dy$$

This cancels out due to symmetry

$$= \frac{10^{-8}}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{2,0,-2}{(y'^2+8)^{3/2}}\,dy'.$$

$$= \frac{10^{-8}\times 2}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{dy'}{(y'^2+8)^{3/2}}\,\hat{a}_x - \frac{10^{-8}\times 2}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{dy'}{(y'^2+8)^{3/2}}\,\hat{a}_z$$

$$= \frac{10^{-8}\times 2}{2\pi\epsilon_0}\int_{0}^{\infty} \frac{dy'}{(y'^2+8)^{3/2}}\,\hat{a}_x - \frac{10^{-8}\times 2}{2\pi\epsilon_0}\int_{0}^{\infty} \frac{dy'}{(y'^2+8)^{3/2}}\,\hat{a}_z$$

$$= \cancel{\frac{10^{-8}\times 2}{2\pi\epsilon_0}}$$

Let, $y' = \sqrt{8}\tan\theta$

$y'^2 + 8 = 8\sec^2\theta$

$dy' = \sqrt{8}\sec^2\theta\, d\theta$

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$$\int_0^{\infty} \frac{dy'}{(y'^2+8)^{3/2}} = \int_0^{\pi/2} \frac{\sqrt{8}\sec^2\theta \, d\theta}{(8\sec^2\theta)^{3/2}}$$

When, $y'=0$, $\theta=0$
$y'=\pi/2$, $\theta=\pi/2$

$$= \frac{1}{8}\int_0^{\pi/2} \cos\theta \, d\theta = \frac{1}{8}\left[\sin\theta\right]_0^{\pi/2} = \frac{1}{8}.$$

$$\therefore \bar{E}_L = \frac{10^{-8}\times 2}{2\pi\epsilon_0}\cdot\frac{1}{8}\hat{a}_x - \frac{10^{-8}\times 2}{2\pi\epsilon_0}\cdot\frac{1}{8}\hat{a}_z$$

$$= \frac{10^{-8}}{8\pi\epsilon_0}\hat{a}_x - \frac{10^{-8}}{8\pi\epsilon_0}\hat{a}_z.$$

Electric field at origin due to surface charge ρ_s be $\bar{E}_s$

$$\bar{E}_s = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_s(\bar{r}-\bar{r}_s)}{|\bar{r}-\bar{r}_s|^3} ds'$$

$ds' = dx'dy'$

$$= \frac{\rho_s}{4\pi\epsilon_0}\iint \frac{(-x', -y', 2)}{\left[x'^2+y'^2+4\right]^{3/2}} dx'dy'$$

Due to symmetry, $\bar{E}_s$ can be simplified as $\rightarrow$

$$\bar{E}_s = \frac{4\times 10^{-9}}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(0,0,2)}{(x'^2+y'^2+4)^{3/2}} dx'dy'.$$

$$\bar{E}_s = \frac{4\times10^{-9}}{4\pi\epsilon_0}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{2\,dy'}{(4+x^2+y^2)^{3/2}}\,\hat{a}_z$$

$$= \frac{4\times10^{-9}}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{2\,dx'\cdot 2}{4+x^2}\,\hat{a}_z$$

$$= \frac{4\times10^{-9}\times 4}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{dx'}{4+x^2}\,\hat{a}_z$$

$$= \frac{4\times10^{-9}}{4\pi\epsilon_0}\times 8\int_{0}^{\infty} \frac{dx}{4+x^2}\,\hat{a}_z$$

$$= \frac{8\times10^{-9}}{2\pi\epsilon_0}\left[\tan^{-1}\frac{x}{2}\right]_0^{\infty} = \cancel{\left(\frac{2\times10^{-9}}{\pi\epsilon_0}\cdot\frac{\pi}{8}\right)\hat{a}_z}$$

$$= \frac{8\times10^{-9}}{2\pi\epsilon_0}\cdot\frac{\pi}{2}\,\hat{a}_z.$$

$$= \frac{2\times10^{-9}}{\epsilon_0}\,\hat{a}_z.$$

Thus, $\bar{E}(0,0,0) = \bar{E}_L + \bar{E}_s = \frac{10^{-8}}{8\pi\epsilon_0}\hat{a}_x + \left(\frac{2\times10^{-9}}{\epsilon_0} - \frac{10^{-8}}{8\pi\epsilon_0}\right)\hat{a}_z$. V/m

$$= 44.95\,\hat{a}_x + 180.94\,\hat{a}_z \text{ V/m}.$$