# EE530 — Handout # 1

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## 1 Levinson-Durbin Algorithm

Let $r(n)$ be the autocorrelation of some sequence and define the $m+1 \times m+1$ (autocorrelation) matrix

$$R(m) = \begin{bmatrix} r(0) & r(1) & \cdots & r(m) \\ r(1) & r(0) & \cdots & r(m-1) \\ \vdots & \vdots & \vdots & \vdots \\ r(m) & r(m-1) & \cdots & r(0) \end{bmatrix}$$

Note that $R(m)$ is symmetric and Toeplitz (i.e., the $(i,j)$ entry of matrix $R(m)$ depends only on $i-j$). We note the following two properties of $R(m)$:

**Property 1:**

$$R(m+1) = \left[\begin{array}{cccc|c} & & & & r(m+1) \\ & R(m) & & & r(m) \\ & & & & \vdots \\ & & & & r(1) \\ \hline r(m+1) & r(m) & \cdots & r(1) & r(0) \end{array}\right]$$

**Property 2:** Let $J(m)$ be an $m+1 \times m+1$ matrix with ones on the non-principal diagonal, zeros everywhere else:

$$J(m) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Note that pre-multiplying by $J(m)$ reverses the order of rows, whereas post-multiplying by $J(m)$ reverses the order of columns. It is easy to verify that

$$R(m)J(m) = J(m)R(m).$$

Suppose $a^{(m)} = [\ 1 \quad a_m(1) \quad \cdots \quad a_m(m)\ ]^T$ is the solution to the order $m$ FIR whitening problem, so that

$$R(m)a^{(m)} = \begin{bmatrix} \alpha_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

To obtain a solution to the order $m+1$ FIR whitening problem, consider (see property 1)

$$R(m+1)\begin{bmatrix} a^{(m)} \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_m \\ 0 \\ \vdots \\ 0 \\ \gamma_m \end{bmatrix} \tag{1}$$

where

$$\gamma_m = \sum_{k=0}^{m} r(k+1)a_m(m-k).$$

Multiply (1) on the left by $J(m+1)$ to get

$$J(m+1)R(m+1)\begin{bmatrix} a^{(m)} \\ 0 \end{bmatrix} = J(m+1)\begin{bmatrix} \alpha_m \\ 0 \\ \vdots \\ 0 \\ \gamma_m \end{bmatrix}$$

which gives

$$R(m+1)\begin{bmatrix} 0 \\ \tilde{a}^{(m)} \end{bmatrix} = \begin{bmatrix} \gamma_m \\ 0 \\ \vdots \\ 0 \\ \alpha_m \end{bmatrix} \tag{2}$$

where $\tilde{a}^{(m)} = J(m)a^{(m)} = [\ a_m(m) \quad \cdots \quad a_m(1) \quad 1\ ]^T$ is the reversal of $a^{(m)}$. Multiplying (2) by $c_{m+1} = -\frac{\gamma_m}{\alpha_m}$ and adding it to (1) gives

$$R(m+1)a^{(m+1)} = \begin{bmatrix} \alpha_m \\ 0 \\ \vdots \\ 0 \\ \gamma_m \end{bmatrix} - \frac{\gamma_m}{\alpha_m}\begin{bmatrix} \gamma_m \\ 0 \\ \vdots \\ 0 \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \alpha_{m+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$a^{(m+1)} = \begin{bmatrix} a^{(m)} \\ 0 \end{bmatrix} + c_{m+1}\begin{bmatrix} 0 \\ \tilde{a}^{(m)} \end{bmatrix}$$

and

$$\alpha_{m+1} = \alpha_m - \frac{\gamma_m^2}{\alpha_m} = \alpha_m(1 - c_{m+1}^2).$$

The $c_m$ are called reflection coefficients.

**Algorithm:**
For $m = 0$, we have $R(0)a^{(0)} = \alpha_0$, which gives $a_0(0) = 1$ and $\alpha_0 = r(0)$. For $m = 0, 1, \ldots, n-1$, given $a^{(m)}$, $\alpha_m$, and $r(0), r(1), \ldots, r(m+1)$

- Obtain $\gamma_m = \sum_{k=0}^{m} r(k+1)a_m(m-k)$.

- Obtain $c_{m+1} = -\gamma_m/\alpha_m$.
- Update $\alpha_m$

$$\alpha_{m+1} = \alpha_m(1 - c_{m+1}^2).$$

- Update $a^{(m)}$

$$a^{(m+1)} = \begin{bmatrix} a^{(m)} \\ 0 \end{bmatrix} + c_{m+1} \begin{bmatrix} 0 \\ \tilde{a}^{(m)} \end{bmatrix}$$

It is easy to verify that the above update from order $m$ to order $m+1$, involve $2m+3$ multiplications and an equal number of additions for. Hence the total number of operations to obtain the order $n$ FIR whitening filter solution is $\sum_{m=0}^{n-1}(2m+3) = n(n+2) = O(n^2)$. This is to be compared with the $O(n^3)$ operations required to solve an $n \times n$ system of linear equations.

## 2 Cholesky (triangular) decomposition of $R(n)$

From the normal equations for the FIR whitening problem of order $n$ we have

$$R(n)a^{(n)} = \begin{bmatrix} \alpha_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As before, premultiplying by $J(n)$ on both sides,

$$J(n)R(n)a^{(n)} = R(n) \begin{bmatrix} a_n(n) \\ \vdots \\ a_n(1) \\ 1 \end{bmatrix} = J(n) \begin{bmatrix} \alpha_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_n \end{bmatrix}.$$

Therefore, using the normal equations for order $0, 1, \ldots, n$ we have

$$R(n) \begin{bmatrix} 1 & a_1(1) & \cdots & a_{n-1}(n-1) & a_n(n) \\ 0 & 1 & \cdots & a_{n-1}(n-2) & a_n(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n(1) \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 & \cdots & 0 & 0 \\ X & \alpha_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X & X & \cdots & \alpha_{n-1} & 0 \\ X & X & \cdots & X & \alpha_n \end{bmatrix}$$

which can be represented as

$$R(n)T = \tilde{L},$$

where $T$ is an upper triangular matrix (with ones on the principal diagonal), whose columns are solutions to the normal equation (of FIR whitening problem) of order $0, 1, \ldots, n$, and $\tilde{L}$ is a lower triangular matrix with $\alpha_0, \ldots, \alpha_n$ on its principal diagonal. Therefore,

$$T^T R(n) T = T^T \begin{bmatrix} \alpha_0 & 0 & \cdots & 0 & 0 \\ X & \alpha_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X & X & \cdots & \alpha_{n-1} & 0 \\ X & X & \cdots & X & \alpha_n \end{bmatrix}.$$

Observe that the right hand side of the above equation is a product of lower triangular matrices, and hence is lower triangular, whereas the left hand side is symmetric. This implies that the right hand side is a diagonal matrix, say $D$. Obviously, the diagonal elements of $D$ are $\alpha_0, \ldots, \alpha_n$. In other words,

$$T^T R(n) T = D = \text{diagonal}\{\alpha_0, \ldots, \alpha_n\},$$

which gives $R(n) = (T^T)^{-1} D T^{-1} = (D^{1/2} T^{-1})^T (D^{1/2} T^{-1}) = L L^T$ where $L = (D^{1/2} T^{-1})^T$ is a lower triangular matrix. This is called the Cholesky decomposition or Cholesky factorization of $R(n)$.

# 3 References


- Text: page 520 "Lattices and Levinson Algorithm" and appendix 7A.
- Proakis: Sections 11.2 and 11.3 (page 857-868).